# NOTE ON THE CONVERSE OF FABRY'S GAP THEOREM

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The gap theorem of Fabry states that if $f(z)=\sum a_k z^{n_k}$ is a power series whose circle of convergence is the unit circle and $\lim n_k/k=\infty$ then the unit circle is the natural boundary of $f(z)$.

Pólya[1] proved the following converse of this result: Let $n_k$ be a sequence of integers for which $\liminf n_k/k<\infty$; then there exists a power series $\sum a_k z^{n_k}$ whose circle of convergence is the unit circle and for which the unit circle is not the natural boundary.

Pólya's result shows that in some sense Fabry's result is the best possible. Perhaps the following direct and elementary proof might be of some interest.

There clearly exist two sequences of integers $u_i$ and $v_i$ such that $v_i=[(1+c_1)u_i]$, $u_{i+1}>v_i^2$, and the number of $n_k$ in $(u_i, v_i)$ is greater than $c_2(v_i-u_i)>c_3u_i$. (The $c$'s denote absolute positive constants.) The existence of these sequences is immediate from $\liminf n_k/k<\infty$. Denote the $n_k$ in the intervals $(u_i, v_i)$ by $n_k'$. We clearly have $\liminf n_k'/k<\infty$. For construction of $f(z)$ we shall use only the $n_k'$. Put $f(z)=\sum a_k z^{n_k'}$. We shall determine the $a_k$ so that the unit circle will be the circle of convergence and the point 1 will be a regular point of $f(z)$. It will suffice to show that there exists a number $l$, $1>l>0$, such that the circle of convergence of

$$f(z+l)=\sum_k a_k(z+l)^{n_k'}=\sum_m b_m z^m$$

has radius greater than $1-l$. We shall choose $l=(r-1)/r$, $r$ a sufficiently large integer. We have by the binomial expansion

$$\sum_k a_k\left(z+\frac{r-1}{r}\right)^{n_k'}=\sum_m z^m\sum_k a_k C_{n_k',m}\left(\frac{r-1}{r}\right)^{n_k'-m}=\sum_m b_m z^m.$$

We have to show that $\limsup b_m^{1/m}<r$, for some choice of the $a_k$ with $\limsup |a_k|^{1/n_k}=1$.

Let $\epsilon$ be a small but fixed number; we distinguish two cases. In case (i), $m$ does not lie in any of the intervals $((u_i/r)(1-\epsilon), (v_i/r)(1+\epsilon))$. Then we show that for every choice of the $a_k$ with $|a_k|\leqq 1$, $\limsup b_m^{1/m}<r-\delta$, $\delta=\delta(\epsilon)$. This means that if $m$ is large enough and does not lie in $((u_i/r)(1-\epsilon), (v_i/r)(1+\epsilon))$ then $b_m<(r-\delta)^m$. Clearly

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(1) G. Pólya, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 65–71.

$$b_m \leqq \sum_k C_{n_k',m}\left(\frac{r-1}{r}\right)^{n_k'-m}. \tag{1}$$

If we define

$$C_{n,m}((r-1)/r)^{n-m} = A_n$$

we find

$$A_{n+1}/A_n = ((r-1)/r)(n+1)/(n-m+1)). \tag{2}$$

By studying the quotient (2) we see that max $A_n = A_{rm} = C_{rm,m}((r-1)/r)^{(r-1)m}$, and by applying Stirling's formula

$$n! \sim (2\pi)^{-1/2} n^{n+1/2} e^{-n}$$

we note that $A_{rm}^{1/m} \to r$ as $m \to \infty$. It follows from (2) that there exists $\eta = \eta(\epsilon) > 0$ such that

$$\begin{aligned} A_{n+1}/A_n &> 1 + \eta \quad \text{for} \quad n < rm/(1+\epsilon) \\ A_{n+1}/A_n &< 1 - \eta \quad \text{for} \quad n > rm/(1-\epsilon) \end{aligned} \tag{3}$$

and hence a simple calculation shows that there exists a $\lambda = \lambda(\epsilon) > 0$ such that

$$A_n < (r - \lambda)^m \tag{4}$$

for $n$ not in $(rm/(1+\epsilon), rm/(1-\epsilon))$.

Now clearly

$$b_m = \sum_1 + \sum_2$$

where in $\sum_1$ the summation is extended over the $n < rm/(1+\epsilon)$ and in $\sum_2$ over the $n > rm/(1-\epsilon)$. (By assumption $m$ does not lie in $((u_i/r)(1-\epsilon), (v_i/r)(1+\epsilon))$ and in (1) the $n_k'$ are all in $(u_i, v_i)$; thus if $m < (u_i/r)(1-\epsilon)$, $n_k' > rm/(1-\epsilon)$ and if $m > (v_i/r)(1+\epsilon)$, $n_k' < rm/(1+\epsilon)$.) Thus from (3) and (4) (by summing a geometric series)

$$b_m < c_4(r-\lambda)^m$$

or

$$\limsup b_m^{1/m} < (r - \delta)$$

which completes the proof.

In case (ii)

$$(u_i/r)(1-\epsilon) < m < (v_i/r)(1+\epsilon) \quad \text{for some } i.$$

We write

$$b_m = b_m' + b_m'',$$

where

$$b'_m = \sum_1 a_k C_{n'_k,m}\left(\frac{r-1}{r}\right)^{n'_k-m}, \qquad b''_m = \sum_2 a_k C_{n'_k,m}\left(\frac{r-1}{r}\right)^{n'_k-m},$$

$\sum_1$ indicates that the summation is extended only over those $k$ for which $n'_k$ does not lie in $(u_i, v_i)$, and in $\sum_2$ the summation is extended over the other $k$. We have

$$b'_m \leqq \sum_1 C_{n'_k,m}\left(\frac{r-1}{r}\right)^{n'_k-m}$$

and we can show that $\limsup b'^{1/m}_m < r$ as before.

Now we show that we can choose the $a_k$ to be such as to make all the $b''_m$ for $(u_i/r)(1-\epsilon) \leqq m \leqq (v_i/r)(1+\epsilon)$ equal to 0. Thus we must determine the $a_k$ so that

$$\sum_2 a_k C_{n'_k,m}\left(\frac{r-1}{r}\right)^{n'_k-m} = 0.$$

These are homogeneous equations for the $a_k$. The number of these equations is less than $2(v_i-u_i)/r$ for sufficiently small $\epsilon$ and the number of unknowns is greater than $c_3(v_i-u_i)$ which is greater than the number of equations for large enough $r$. Thus the system of equations always has a solution, and further we can suppose that the absolute value of the largest $a_k$ is 1. This will insure that the circle of convergence of $f(z)$ will be the unit circle, which completes the proof.

It would be easy to construct by the same method an $f(z)$ whose circle of convergence is the unit circle and whose regular points are everywhere dense on the unit circle.

Let $n_1 < n_2 < \cdots$ be a sequence whose maximal density is $\alpha$. Pólya proved(2) that if $f(z) = \sum a_k z^{n_k}$ is a power series, the radius of convergence of which is 1, then no arc greater than $2\pi\alpha$ of the unit circle is free of singular points. This is clearly a generalization of Fabry's gap theorem. It would be interesting to investigate the converse problem, that is, if $n_1 < n_2 \cdots$ is a sequence of maximal density $\alpha$, what is the greatest $c$ such that there exists a power series $f(z) = \sum a_k z^{n_k}$ whose radius of convergence is 1, and such that the unit circle has a regular arc of length $2\pi c$. In particular, is Pólya's theorem best possible? So far our estimates of $c$ are very much worse. Pólya(3) defines maximal density as follows: Denote by $N(a, b)$ the number of the $n$'s in $(a, b)$. Then

$$\alpha = \limsup_{c\to 0}\ \limsup_{m\to\infty} \frac{N(m, m(1+c))}{cm}.$$

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(2) G. Pólya, Math. Zeit. vol. 29 (1929) pp. 549–640.
(3) Ibid.