SOME REMARKS ON THE MEASURABILITY OF CERTAIN SETS

PAUL ERDÖS

The present note contains some elementary remarks on sets defined by simple geometric properties. Our main tool will be the Lebesgue density theorem.

First we introduce a few notations: $d(a, b)$ denotes the distance from a to b and $S(x, r)$ the open sphere of center x and radius r. A point x of a set A is said to be of metric density 1 if to every ϵ there exists a δ such that $A \cap S(x, r)$, $r < \delta$, has measure greater than $(1-\epsilon)$ times the volume of $S(x, r)$. $\bar{A}$ denotes the closure of A.

(1) Let E be any closed set in n-dimensional euclidean space. Denote by E_r the set of points whose distance from E is r $(r>0)$. We shall prove that E_r has measure 0.

The set E_r is clearly closed and therefore measurable. If it had positive measure it would contain a point of metric density 1. Let x be any point of E_r and $y \in E$ be one of the points in E at distance r from x. Then $S(y, r)$ cannot contain any point of E_r. Thus x cannot be a point of metric density 1, which completes the proof. This proof is due to T. Radó.

(2) Let A be any set of measure 0 on the positive real axis. Denote by E_A the set of points whose distance from E is in A. We shall show that E_A has measure 0. As is well known A is contained in a G_δ, say G of measure 0. Thus it suffices to show that E_g has measure 0. E_g is clearly a G_δ and thus measurable, so that again it will suffice to show that E_g has no point of metric density 1. Let x be any point of E_g and y any one of the points of E closest to it. Denote by $C_x(\eta_1, \eta_2)$ the half cone defined as follows: $z \in C_x(\eta_1, \eta_2)$ if $d(z, x) < \eta_1$ and the angle zxy is less than η_2. Let R be any ray in C_x from x. Denote by z a variable point of R. We assert that if η_1 and η_2 are sufficiently small, $d(z, E)$ is a decreasing function of $d(z, x)$ for which the upper limit of the difference quotient with respect to $d(z, x)$ is less than $-\delta$, with some $\delta > 0$. Let $y_1 \in E$ be one of the points closest to z in E. We assert that $d(y, y_1)$ is small if η_2 is small. Clearly by definition y_1 is contained in $\{S(z, d(z, y))\}$ but not in $S(x, d(x, y))$. Since $d(x, z) < \eta_1$ the difference of these two spheres has small diameter if η_2 is small, which shows that $d(y, y_1)$ is small. Now it is geometrically clear that for sufficiently small η_1, η_2 there exists a $\delta > 0$ such that the upper limit of the difference quotient of $d(z, y_1)$ with respect to $d(z, x)$ is less

Received by the editors May 16, 1945, and, in revised form, June 13, 1945.

than $-\delta$. A fortiori the upper limit of the difference quotient of $d(z, E)$ with respect to $d(z, x)$ is less than $-\delta$. Thus it follows that the set of points on R for which $d(z, E)$ is in A is of measure 0. Thus by a trivial modification of Fubini's theorem we obtain that $C_x \cap E_g$ has measure 0. Thus x could not have been a point of metric density 1, which completes our proof.

Let S be any measurable set on the positive real axis; it is easy to see that E_S is also measurable. For S can be written as $F+A$ where F is an F_σ and A is of measure 0. Now clearly $E_S = E_F + E_A$. E_F is measurable since it is also an F_σ and E_A is of measure 0. Therefore, E_S is measurable.

(3) Denote by M the set of those points for which there is more than one closest point in E. It is known that the necessary and sufficient condition for E to be convex is that M be empty. We shall prove that M has measure 0.

For $x \in M$, denote by $\phi(x)$ the set of points closest to x. Clearly the set of points M_c for which the diameter of $\phi(x)$ is not less than c is closed, thus M is an F_σ and thus measurable. It suffices to show that M_c has measure 0, or that it can have no point of metric density 1. Let $y \in \phi(x)$ be arbitrary ($\phi(x)$ is of course closed). Define $C_x(\eta_1, \eta_2)$ as in (2). We shall prove that no point of $C_x(\eta_1, \eta_2)$ (except x) belongs to M_c and this will show that x cannot have metric density 1. If $z \in M_c \cap C_x(\eta_1, \eta_2)$ there exists a sphere $S(z, r)$, $r \leqq d(z, y)$, such that $S(z, r)$ contains no points of E in its interior and $\{S(z, r)\}$ contains two points u and v of E with $d(u, v) \geqq c$. But u and v cannot be in the interior of $S(x, d(x, y))$. Hence they must be in $[\text{Comp}\,(S(x, d(x, y))]$ $\cap \{S(z, r)\}$ (Comp A denotes the complement of A), but for $\eta_2 = \eta_2(c)$ small enough the diameter of this set is less than c, which is a contradiction. This completes the proof.

The problems in (1) and (3) were suggested to me by Deane Montgomery.

(4) Let x be any point in the complement of E. As before we denote by $\phi(x)$ the set of points in E closest to x. We shall prove that $\sum_{x \notin E} \phi(x)$ has measure 0.

It will be sufficient to prove that no point $z \in \sum_{x \notin E} \phi(x)$ has upper metric density 1.[1] If $z \in \phi(x)$ then $S(x, d(x, z))$ contains no point of E in its interior (and $\sum_{x \notin E} \phi(x) \subset E$), which proves our theorem.

(5) Denote by M_k the set of points for which $\phi(x)$ contains k points not all in a $(k-2)$-dimensional euclidean subspace. In (3) we proved that M_2 has n-dimensional measure 0. I conjecture that M_k

[1] Let E be any set. Then the upper metric density is 1 at almost all points of E. (See, for example, Hildebrandt, Bull. Amer. Math. Soc. vol. 32 (1926) p. 451.)

has Hausdorff dimension $n+1-k$. At present I can prove this only for $k=n+1$. In fact we shall prove that M_{n+1} is denumerable. For the sake of simplicity we shall restrict ourselves to $n=2$. The proof for the general case is not an easy generalization of the case $n=2$, but we omit details.

Suppose then that M_3 is nondenumerable. Then it must contain a point of condensation, x say. Put $r=d(x, E)$. There exist nondenumerably many points z such that $r-\epsilon<d(z, E)<r+\epsilon$, and $S(z, d(z, E))$ contains at least three points of E on its boundary. $S(z, d(z, E))\cap E$ is closed. Denote by t_z the maximum of the smallest side of all possible triangles formed from points of $S(z, d(z, E))\cap E$. By a well known argument there exists a constant $c>0$ such that for every $\delta>0$ there are uncountably many points z satisfying

$$(1) \qquad d(z, x) < \epsilon, \qquad c \leqq t_z < c + \delta.$$

Choose δ small, and consider $\bigcup_z S(z, d(z, E))$ with z satisfying (1). Denote the boundary of this domain by B. Let p be any point of B and denote by $C_p(\eta)$ the half cone whose vertex is at p and whose center line is the extension of the line from x to p. It is easy to see that for sufficiently small ϵ there exists an $\eta>0$ such that for any point p on B, $C_p(\eta)$ does not contain any point of B other than p. From this it can be shown by straightforward methods that B is a rectifiable curve[2] and hence can contain only countably many arcs of circles. This we shall show to be false. Let z_1 be any point satisfying (1). Denote by (a, b) the arc on $S(z_1, d(z_1, E))$ determined by the side of length t_{z_1}. Since we can choose z_1 in uncountably many ways, we can assume that z_1 has been chosen so that the arc (a, b) does not lie on B. But since $a\in B$ and $b\in B$ there must exist a point z_2 satisfying (1) such that $S(z_2, (d(z_2, E))$ intersects $S(z_1, d(z_1, E))$ in two points u and v on the arc (a, b). Therefore if δ is a sufficiently small fraction of c,

$$t_{z_2} < c$$

which shows that z_2 does not satisfy (1), an evident contradiction. This completes the proof.

(6) In (1) we proved that E_r has n-dimensional measure 0. Let us now assume that E is bounded, then we shall sketch a proof of the fact that E_r has finite $(n-1)$-dimensional measure.

Let D be the diameter of E. Assume first that r is large. Let x be a fixed point of E and p any point of E_r. Then it is easy to see that $C_p(\eta)$

[2] Pauc, J. Reine Angew. Math. vol. 185 (1943) pp. 127–128. Pauc proves a more general theorem.

does not contain any point of E_r other than p. $C_p(\eta)$ is defined as in (5). From this it can be shown that E_r has finite $(n-1)$-dimensional measure. Let us not assume now that r is large. We then write $E=\bigcup_{k=1}^m E^{(k)}$ where the E^k's are closed and their diameter is less than ϵ. Then, by what has been shown before, if ϵ is small enough $E_r^{(k)}$ has finite $(n-1)$-dimensional measure. Clearly $E_r \subset \bigcup_{k=1}^m E_r^{(k)}$. But E_r is closed, therefore its $(n-1)$-dimensional measure exists, and it clearly can not be 0, since it separates the space. Thus E_r must have finite $(n-1)$-dimensional measure.

Added in proof. The author has recently discovered that the following two theorems have been stated by C. Pauc, Revue Scientifique vol. 77 (1939) no. 8: Let the set E be in the plane then M_2 is contained in the sum of countably many Jordan curves and M_3 is countable.

UNIVERSITY OF MICHIGAN