ON THE HAUSDORFF DIMENSION OF SOME SETS
IN EUCLIDEAN SPACE

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Let $E$ be a closed set in $n$-dimensional space, $x$ a point not in $E$. Denote by $S(x)$ the largest sphere of center $x$ which does not contain any point of $E$ in its interior. Put $\phi(x) = E \cap S(x)$. ($\bar{A}$ denotes the closure of $A$.) Denote by $M_k$ the set of points for which $\phi(x)$ contains $k$ or more linearly independent points (that is, points which do not lie in any $(k-2)$-dimensional hyperplane). $M_k$ is defined for $k \leq n+1$. In a previous paper I proved that $M_2$ has $n$-dimensional measure 0 and conjectured that $M_k$ has Hausdorff dimension not greater than $n+1-k$. In the present note we shall prove this conjecture. In my previous paper I also proved that $M_{n+1}$ is countable, but the proof there given applied only for the case $n=2$; now we are going to give a general proof.

Let $R$ be any set in $n$-dimensional space. Let $x \in R$. We define the contingent\(^1\) of $R$ at $x$ ($\text{contg}_R x$) as follows: The contingent will be a subset of the unit sphere. A point $z$ of the unit sphere belongs to $\text{contg}_R x$ if and only if there exists a sequence of points $y_1, y_2, \ldots$ in $R$ converging to $x$ so that the direction of the vector connecting $x$ with $y_1$ tends to the direction of the vector connecting the center of the unit sphere with $z$. First we state the following lemma.

**LEMMA.** Let there be given a set $R$ in $n$-dimensional space. Assume that for every $x$, $\text{contg}_R x$ does not contain any point of the intersection of the unit sphere with a $k$-dimensional hyperplane going through its center (the hyperplane can depend on $x$). Then $R$ is contained in the sum of countably many surfaces of finite $(n-k)$-dimensional measure.

This lemma is well known.\(^2\)

**THEOREM 1.** Let $k < n+1$. Then $M_k$ is contained in the sum of countably many surfaces of finite $(n+1-k)$-dimensional measure. If $k = n+1$, then $M_k$ is countable.\(^3\)

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\(^3\) For $n=2$ this theorem is proved by C. Pauc, Revue Scientifique, August, 1939.
Remark. This clearly means that the Hausdorff dimension of $M_n$ ($k \leq n+1$) is not greater than $n+1-k$.

Let us first consider the case $k = n+1$. Assume that $x \in M_{n+1}$. Let $z_i \in \phi(x)$, $i = 1, 2, \ldots, n+1$, and assume that the $z_i$'s are linearly independent. Denote by $f(x)$ the maximum value of the volume of the simplices determined by the $z_i$'s (since $\phi(x)$ is closed the maximum is attained). Define now $N_{n+1}(c) = N$ to consist of all the points $x \in M_{n+1}$ for which $f(x) \geq c$. It clearly will be sufficient to show that $N$ is countable (for every $c$). In fact we shall show that $N$ is isolated (in other words no $x \in N$ is a limit point of $N - x$), that is, we shall prove that for every $x \in N$ contg $x$ is empty. If this would not hold then $N$ would contain an infinite sequence of points $y_j$ converging to $x$ so that the direction of the line connecting $x$ with $y_j$ would converge to a fixed direction. Let $Z_j$ be a point of $\phi(x)$ which is closest to $y_j$, and let $A_j$ be the (unique) hyperplane through $Z_j$ perpendicular to the segment $xy_j$. It is easy to see that as $j \to \infty$, $A_j$ converges to a limiting hyperplane $A$. Moreover it is easily seen that the set $\phi(y_j)$ is ultimately contained in any preassigned neighborhood of $A$. Thus for large enough $j$, the volume $f(y_j)$ must be less than $c$, an evident contradiction; this completes our proof.

Next we prove our theorem in the general case. Let $k \leq n$ and define $M'_k$ to be the set of all points $x$ for which the maximum number of linearly independent points in $\phi(x)$ is exactly $k$. It will clearly be sufficient to show that $M'_k$ is contained in the sum of countably many surfaces of finite $(n+1-k)$-dimensional measure. Let $x \in M'_k$, and let $f(x)$ be the maximum volume of the $k$-dimensional simplices formed from the points $z_i$, $i \leq k+1$, where $z_i \in \phi(x)$. $x \in M'_k$ if $f(x) \geq c$. Let $x \in N'$, and $z_i$, $i \leq k+1$, be the points which determine a simplex of maximal volume. Then a simple geometrical argument (similar to the previous one) shows that contg $x$ consists only of the directions through $x$ which are perpendicular to the hyperplane determined by the $z_i$'s, $i \leq k+1$. Thus our theorem follows from the lemma.

Let $E$ be a closed set, $x \in E$. Denote by $g(x)$ the distance of $x$ from $E$. It has been proved\(^4\) that $g(x)$ has a derivative $-\cos \alpha$ in every direction $(x, y)$, where $\alpha$ is the smallest angle formed by the direction $(x, y)$ with the direction $(x, z)$, $z$ in $\phi(x)$. Clearly if $x \in E$ the derivative of $g(x)$ can be $0$. We shall show that the derivative of $g(x)$ is $0$ for almost all points of $E$.

Let $x \in E$. Denote by $S(x, \varepsilon)$ the sphere of center $x$ and radius $\varepsilon$. Denote by $G(x, \varepsilon)$ the greatest distance of the points of $S(x, \varepsilon)$ from $E$. We are going to prove the following theorem.

**Theorem 2.** For almost all points of $E$ (that is, for all points of $E$ except a set of $n$-dimensional measure 0)

\[
\lim_{\varepsilon \to 0} G(x, \varepsilon)/\varepsilon = 0.
\]

It is well known that almost all points of $E$ are points of Lebesgue density 1. Let $x$ be such a point, and suppose that

\[
\lim_{\varepsilon \to 0} G(x, \varepsilon)/\varepsilon \neq 0.
\]

This means that there exists an infinite sequence $\varepsilon_i$ and points $z_i$, $z_i \in S(x, \varepsilon_i)$, $\varepsilon_i \to 0$, such that the distance of $z_i$ from $E$ is greater than $c\varepsilon_i$, where $c > 0$. But this clearly means that $x$ can not have Lebesgue density 1. This contradiction establishes our theorem.

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