Let there be given \( n \) elements \( a_1, a_2, \ldots, a_n \). By \( A_1, A_2, \ldots, A_m \) we shall denote combinations of the \( a \)'s. We assume that we have given a system of \( m > 1 \) combinations \( A_1, A_2, \ldots, A_m \) so that each pair \((a_i, a_j)\) is contained in one and only one \( A \). Then we prove

**Theorem 1.** We have \( m \geq n \), with equality occurring only if either the system is of the type \( A_1 = (a_1, a_2, \ldots, a_{n-1}), A_2 = (a_1, a_3), \ldots, A_n = (a_2, a_n) \), or if \( n \) is of the form \( n = k(k-1) + 1 \) and all the \( A \)'s have \( k \) elements, and each \( a \) occurs in exactly \( k \) of the \( A \)'s.

**Corollary:** If the elements \( a_i \) are points in the real projective plane the theorem can be stated as follows: Let there be given \( n \) points in the plane, not all on a line. Connect any two of these points. Then the number of lines in this system is \( \geq n \). In this case equality occurs only if \( n - 1 \) of the points are on a line.

This corollary can be proved independently of Theorem 1 by aid of the following theorem of Gallai (= Grünwald):

Let there be given \( n \) points in the plane, not all on a line. Then there exists a line which goes through two and only two of the points.

**Remark:** The points of inflexion of the cubic show that it is essential that the points should all be real, thus Gallai's theorem permits no projective and a fortiori no combinatorial formulation. Also the result clearly fails for infinitely many points.

We now give Gallai's ingenious proof: Assume the theorem false. Then any line through two of the points also goes through a third. Project one of the points, say \( a_1 \), to infinity, and connect it with the other points. Thus we get a set of parallel lines each containing two or more points \( a_i \) (in the finite part of the plane). Consider the system of lines connecting any two of these points, and assume that the line \((a_i, a_j, a_k)\) forms the smallest angle with the parallel lines. (This line again contains at least three points). But the line connecting \( a_i \) with \( a_1 \) (at infinity) contains at least another (finite) point \( a_r \), and clearly (see figure) either the line \((a_i, a_r)\)
or the line \((a_r, a_k)\) forms a smaller angle with the parallel lines then 
\((a_l a_j a_k)\). This contradiction establishes the result.

**Remark:** Denote by \(f(n)\) the minimum number of lines which go 
through exactly two points. It is not known whether \(\lim f(n) = \infty\). All 
that we can show is that \(f(n) \geq 3\).

Now we prove the corollary as follows: We use induction. Assume that 
the number of lines determined by \(n - 1\) points, not all on a line, is 
\(\geq n - 1\). Then we shall prove that \(n\) points, not all on a line, determine at 
least \(n\) lines.

Let \((a_1, a_2)\) be a line going through two points only. Consider the points 
\(a_3, a_4, \ldots, a_n\). If they are all on a line, then \((a_1, a_i), i = 2, 3, \ldots, n\) and 
\((a_2, a_3, \ldots, a_n)\) clearly determine \(n\) lines. If they do not all lie on a line, 
then they determine at least \(n - 1\) lines, and \((a_1, a_2)\) is clearly not one of 
these lines. Thus together with \((a_1, a_2)\) we again get at least \(n\) lines. The 
same induction argument shows that we get exactly \(n\) lines only if \(n - 1\) 
of the points lie on a line, q.e.d.

**Proof of theorem 1.** For simplicity we shall call the elements 
\(a_1, a_2, \ldots, a_n\) points and the sets \(A_1, A_2, \ldots, A_m\) lines. Denote by \(k_i\) the 
number of lines passing through the point \(a_i\), and by \(s_j\) the number of 
points on the line \(A_j\). We evidently find (by counting the number of 
incidences in two ways)

\[
\sum_{j=1}^{m} s_j = \sum_{i=1}^{n} k_i \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)
\]

Further if \(A_j\) does not pass through \(a_i\), then

\[
s_j \leq k_i \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)
\]

(2) follows from the fact that \(a_i\) can be connected by a line (i.e. an \(A\)) 
to all the \(s_j\) points of \(A_j\), and any two of these lines are different, since 
otherwise they would have two points in common.

Assume now that \(k_n\) is the smallest \(k_i\) and that \(A_1, A_2, \ldots, A_{k_n}\) are the 
lines through \(a_n\). We may suppose that each line contains at least two 
points, since otherwise it could be omitted. Also \(k_n > 1\), for otherwise all 
the points are on a line. Thus we can find points \(a_i\) on \(A_i\), \(a_i \neq a_k\), 
\(i = 1, 2, \ldots, k_n\). Also if \(i \neq j, i \leq k_n, j \leq k_n\) then \(a_i\) is not on \(A_j\) (for
otherwise $A_i$ and $A_j$ would have two points in common). Hence by (2) (putting $k_n = v$)

$s_2 \leq k_1, \ s_3 \leq k_2, \ldots, \ s_{r} \leq k_{r-1}, \ s_1 \leq k_r; \ s_j \leq k_n$ for $j > r$. (3)

From (1), (3) and the minimum property of $k_n$ we obtain $m \geq n$, which proves the first part of Theorem 1.

We now determine the cases where $m = n$. If $m = n$, then all the inequalities of (3) have to be equalities. Consequently we can renumerate the points so that $s_1 = k_1, s_2 = k_2, \ldots, s_n = k_n$. We may suppose that $k_1 \geq k_2 \geq \ldots \geq k_n > 1$. There are two cases:

a) $k_1 > k_2$. Hence by $s_1 = k_1 > k_i (2 \leq i \leq n)$, (2) shows that all the $a_i (i \geq 2)$ lie on $A_1$. Of course $a_1$ does not lie on $A_1$ and we have the first case of Theorem 1.

b) $k_1 = k_2$. If no $k_i$ is less than $k_1$ then clearly $k_i = s_j (1 \leq i, j \leq n)$. We shall show that this is the only possibility. If $k_j < k_1$, then we have by (2) that $a_j$ lies on both $A_1$ and $A_2$. Hence $k_n$ is the only $k$ which can be less than $k_1$. Now $s_n = k_n$ different lines contain $a_n$. Any line through $a_n$ contains one further point and all but one contain two further points, since $k_1 = k_2 = \ldots = k_{n-1} > k_2 \geq 2$. Thus there are at least two lines which do not contain $a_n$; for both of these lines we have by (2) $s_j \leq k_n$. This contradicts $s_1 = s_2 = \ldots = s_{n-1} > k_n$.

Apart from case a) we only have the case where $s_i = k_j = k, (1 \leq i, j \leq n)$. It is easily seen that then $n = k (k - 1) + 1$, and also that any pair of lines has exactly one intersection point. For if $A_i$ does not intersect $A_j$; and if $a_i$ lies on $A_i$ then we infer from (2) that $k_i \geq s_i + 1$ which is not possible since $k_i = s_i = k$. The two dimensional projective finite geometries with $k - 1 = p^2, p$ prime, are known to be systems of this type, but F. W. Levi\thinspace 3) constructed a non-projective example with $k = 9$.