THE SET ON WHICH AN ENTIRE FUNCTION IS SMALL.*

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Let \( f(z) \) be an entire function and \( M(r) \) the maximum of \( |f(z)| \) on \( |z| = r \). We give some results on the density of the set of points at which \( |f(z)| \) is small in comparison with \( M(r) \); although simple, these results seem not to have been noticed before.

If \( E \) is a measurable set in the \( z \)-plane, we denote by \( D_R(E) \) the ratio \( m(z \in E, |z| \leq R)/\pi R^2 \) and by \( \overline{D}(E) \) and \( \underline{D}(E) \) the upper and lower densities of \( E \), that is the superior and inferior limits of \( D_R(E) \) as \( R \to \infty \). For a fixed function \( f(z) \), let \( E_\lambda \) be the set of points \( z \) for which \( \log |f(z)| = (1 - \lambda) \log M(\|z\|) \).

**Theorem 1.** For any \( \lambda > 1 \), there is a number \( K \), the same for all functions \( f(z) \), such that \( \overline{D}(E_\lambda) \leq K \). Moreover, \( 0 < K \leq \lambda^{-1} \).

In particular, for \( \lambda = 2 \), the upper density of the set where \( |f(z)| \leq 1/M(\|z\|) \) is at most \( 1/2 \). Much stronger results are known for entire functions of small finite order. The interest of Theorem 1 is that it holds for all entire functions and that, contrary to what might be expected, \( K \) is strictly positive. We shall show that a lower bound on \( K \) is given by \( \frac{62}{1 + \delta} \) where \( \delta \) is the positive root of \( \delta(2 + \delta)^{\lambda^{-1}} = 1 \). For \( \lambda = 2 \), this can be improved to \( \lambda^{-1} \); the same method will yield better values for other choices of \( \lambda \). For lower density, the following is true.

**Theorem 2.** As \( \lambda \to \infty \), \( \overline{D}(E_\lambda) = o(\lambda^{-1}) \).

It might be conjectured that this also holds for the upper density, and for the numbers \( K = K(\lambda) \).

We first prove that \( \lambda^{-1} \) is an upper bound for \( \overline{D}(E_\lambda) \). Consider the integral

\[
I = \frac{1}{2 \pi} \int_0^{2 \pi} [\log M(r) - \log |f(re^{i\theta})|] d\theta.
\]

Let \( f(z) = z^p g(z) \), \( g(0) \neq 0 \). Then, by Jensen's theorem.

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\[ I = \log M(r) - p \log r - (1/2\pi) \int_{0}^{2\pi} \log |g(re^{i\theta})| \, d\theta \]
\[ \leq \log M(r) - p \log r - \log |g(0)|. \]

Let \( H_{r,\lambda} \) be the set of values of \( \theta \) for which \( \log |f(re^{i\theta})| \) is less than \((1 - \lambda) \log M(r)\). By applying to the integral \( I \) the identity \( \int \phi(x) \, dx = \int \psi(r) \, dr \) where \( \phi(x) \geq 0 \) and \( \psi(r) \) is the measure of the set on which \( \phi(x) \geq r \), the integral \( I \) may also be expressed as

\[ I = (2\pi)^{-1} \log M(r) \int_{0}^{\infty} m(H_{r,\lambda}) \, d\lambda. \]

Hence, writing \( C = \log |g(0)| \), we have

\[ (1/2\pi) \int_{0}^{\infty} m(H_{r,\lambda}) \, d\lambda \leq 1 - p \log r + C. \]

Choose \( R_0 \) so that \( M(r) > 1 \) for \( r \geq R_0 \); then

\[ m(z \in E, R_0 \leq |z| \leq R) = \int_{R_0}^{R} m(H_{r,\lambda}) r \, dr. \]

Integrating this with respect to \( \lambda \) and using (1), we have

\[ \int_{0}^{\infty} D_{E_{\lambda}}(E_{\lambda}^{*}) \, d\lambda \leq 1 - R_0^2/R^2 - (\log M(r)) \int_{R_0}^{R} (p \log r + C) r \, dr \]

where \( E_{\lambda}^{*} \) is \( E_{\lambda} \) with the circle \(|z| \leq R_0 \) deleted.

We may suppose that \( f(z) \) is not a polynomial. (In this case, it is easily seen that \( D(E_{\lambda}) = 0 \) for all \( \lambda > 0 \).) Since \( \log M(r) \) is convex in \( \log r \), it follows that \( \log r = o(\log M(r)) \) as \( r \) tends to infinity, and hence that the right side of (2) is \( 1 + o(1) \) as \( R \to \infty \). As \( \lambda \) increases, the sets \( E_{\lambda}^{*} \) decrease and \( D_R(E_{\lambda}^{*}) \) is monotone for fixed \( R \). Thus, \( \lambda D_R(E_{\lambda}^{*}) \leq \int_{0}^{\infty} D_{E_{\lambda}}(E_{\lambda}^{*}) \, d\lambda \) and letting \( R \) increase, we have \( \lambda D(E_{\lambda}) = \lambda D(E_{\lambda}^{*}) \leq 1 \).

The proof of Theorem 2 also falls out of the inequality (2). Letting \( R \) tend to infinity, we have

\[ \int_{0}^{\infty} D(E_{\lambda}) \, d\lambda \leq 1 \]

and since the integrand is monotonic, \( \lim_{\lambda \to \infty} \lambda D(E_{\lambda}) = 0 \).

To obtain a lower bound on \( K \), the least upper bound of \( D(E_{\lambda}) \) for all functions \( f(z) \), we investigate a special function. Consider the product
\[ f(z) = \prod_{n=1}^{\infty} (1 - z/a^n)^{a^n}, \quad a > b > 1, \]

which defines an entire function of order \( \log b/\log a \). Put

\[ \phi(z) = |f(z)| M(r)^{\lambda-1} = \prod_{k=1}^{\infty} \left\{ 1 - \frac{z/a^k}{(1 + r/a^k)^{\lambda-1}} \right\}^{a^k}. \]

Suppose that \( z \) lies in the region \( S \) described by

\[ |1 - z/a^n| (1 + r/a^n)^{\lambda-1} \leq \beta < 1. \]

Let \( r/a^n \) be less than \( \gamma \) for all \( z \) in \( S \). Then,

\[ \phi(z) \leq \prod_{k<n} (1 + r/a^k)^{\lambda a^k} \prod_{k=n} \left( 1 - z/a^k \right) \left( 1 + r/a^k \right)^{\lambda a^k} \leq (\lambda \gamma a^{n-1})^{\lambda b} (\lambda \gamma a^{n-2})^{\lambda b^2} \cdots (\lambda \gamma a)^{\lambda b^{n-2}} \exp \left\{ \lambda \gamma a^{n} \sum (b/a)^k \right\} \]

and

\[ \log \phi(z) \leq b^n \left\{ \frac{\lambda \log \lambda y}{b-1} + \frac{\lambda b \log a}{(b-1)^2} + \frac{\lambda b y}{a-b} + \log \beta \right\}. \]

As \( b \) and \( a \) tend to infinity in such a manner that \( b^{-1} \log a \) and \( b/a \) approach zero (e.g., \( a = b^2 \)), the bracket approaches \( \log \beta \) which is negative. Thus, for any \( \beta < 1 \) and for suitable \( a \) and \( b \), \( \phi(z) < 1 \) for all \( z \) in \( S \), and for the special function that we have constructed, \( S \subseteq E_\lambda \).

There is a set of type \( S \) enclosing each of the points \( z = a^n \). We now estimate the upper density of the union of these sets, and hence the upper density of \( E_\lambda \). We may take \( \beta = 1 \). Put \( w = z/a^n - \rho e^{i\theta} \); the set \( S \) corresponds to the set \( S^* \) bounded by the curve \( |1 - w| (1 + \rho)^{\lambda-2} = 1 \). The circle \( |w - 1| < \delta \) where \( \delta (3 + \delta)^{\lambda-2} = 1 \) lies in \( S^* \). The ratio \( D_{1+\delta}(S^*) \) is at least \( \delta^2/(1+\delta)^2 \) and since this is independent of \( n \), this number is a lower bound for \( K(\lambda) \). A better bound can be obtained by computing the radius \( \rho_0 \) for which \( D_{\rho_0}(S^*) = m(w \in S^*; |w| \leq \rho_0) / \pi \rho_0^2 \) is greatest. This number is then the desired lower bound. In the special case \( \lambda = 2 \), numerical integration gives the value .1925 for this ratio.

With reference to generalizations, we observe that the relations (1) and (2) hold with \( p = 0 \) with any subharmonic function \( v(z) \) replacing the function \( \log |f(z)| \), and with \( \mu(r) = \max_{\delta} v(\rho_0 e^{i\theta}) \) replacing \( \log \beta \), provided that \( C - v(0) \) is finite. In addition, there is equality instead of inequality in (1) and (2) if \( v(z) \) is a harmonic function without singularities.