ON A TAUBERIAN THEOREM CONNECTED WITH THE NEW PROOF OF THE PRIME NUMBER THEOREM

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In our new and elementary proof of the prime number theorem, Selberg and I(1) prove the following Tauberian theorem:

Let $1 < p_1 < p_2 < \ldots$ be an infinite sequence of real numbers which satisfies

$$I. \quad \sum_{p_i \leqslant x} (\log p_i)^2 + \sum_{p_i p_j \leqslant x} \log p_i \log p_j = 2x \log x + o(x \log x)$$

$$II. \quad \sum_{p_i \leqslant x} \frac{\log p_i}{p_i} = [1 + o(1)] \log x$$

$$III. \quad \vartheta(x) = \sum_{p_i \leqslant x} \log p_i > cx.$$

Then

$$\lim_{x \to \infty} \vartheta(x)/x = 1. \tag{1}$$

I is the fundamental asymptotic formula of Selberg(2) for which he obtained an ingenious and elementary proof, and which was the starting point of our investigations. *II* is due to Mertens and *III* to Tchebichef, both of course have well-known elementary proofs.

It might be of interest to investigate whether (1) can be deduced from *I* alone. This indeed turns out to be the case if we use *I* with the stronger error term $O(x)$ (which indeed was proved by Selberg(2)). Thus we shall prove the following

THEOREM 1. *Let* $1 < p_1 < p_2 < \dots$ *Assume that*

$$I'. \quad \sum_{p_i \leqslant x} (\log p_i)^2 + \sum_{p_i p_j \leqslant x} \log p_i \log p_j = 2\,x \log x + O(x).$$

Then (1) *holds.*

We shall prove Theorem 1 by showing that I' implies

$$II'. \quad \sum_{p_i \leqslant x} \frac{\log p_i}{p_i} = \log x + O(1).$$

II and III are easy consequences of II', (thus (1) follows from our work with Selberg[1]). This is clear for II and easy for III, for if III does not hold, we have $\vartheta(x) < \epsilon x$ with $\epsilon < 1$ for suitable x. Then we evidently have

$$\sum_{x\sqrt{\epsilon}}^{x} \frac{\log p_i}{p_i} < \frac{\vartheta(x)}{\epsilon^{1/2} x} < \epsilon^{1/2},$$

thus II' cannot hold[3].

Our principal tool in proving Theorem 1 will be the following Tauberian theorem which is of interest by itself:

THEOREM 2. *Let* $a_k \geqslant 0$, *put* $s_m = \sum_{k=1}^{m} a_k$. *Assume that*

$$S(n) = \sum_{k=1}^{n} a_k (s_{n-k} + k) = n^2 + O(n). \qquad (2)$$

Then

$$s_n = n + O(1).$$

First we shall show that I' implies

$$I'': \quad \sum_{p_i \leqslant x} \frac{(\log p_i)^2}{p_i} + \sum_{p_i p_j \leqslant x} \frac{\log p_i \log p_j}{p_i p_j} = (\log x)^2 + O(\log x).$$ [4]

We use partial summation. Put

$$D(u) = \sum_{p_i \leqslant u} (\log p_i)^2 + \sum_{p_i p_j \leqslant u} \log p_i \log p_j = 2u \log u + O(u).$$

We evidently have

$$A(x) = \sum_{p_i \leqslant x} \frac{(\log p_i)^2}{p_i} + \sum_{p_i p_j \leqslant x} \frac{\log p_i \log p_j}{p_i p_j}$$

$$= \sum_{p_i \leqslant x} \frac{(\log p_i)^2}{[p_i]} + \sum_{p_i p_j \leqslant x} \frac{\log p_i \log p_j}{[p_i p_j]}$$

$$+ O\Big(\sum_{p_i \leqslant x} \frac{(\log p_i)^2}{p_i} + \sum_{p_i p_j \leqslant x} \frac{\log p_i \log p_j}{(p_i p_j)^2} \Big)$$

$$= \sum_{p_i \leqslant x} \frac{(\log p_i)^2}{[p_i]} + \sum_{p_i p_j \leqslant x} \frac{\log p_i \log p_j}{[p_i p_j]} + O(1)$$

(since $D(u) = O(u \log u)$). Thus

$$A(x) = \sum_{u=1}^{x} \frac{D(u+1) - D(u)}{u} + O(1)$$

$$= \sum_{u=2}^{x-1} \frac{D(u)}{u(u-1)} + \frac{D(x)}{x} + O(1) = (\log x)^2 + O(\log x),$$

which proves II'.

Put now

$$e^n = x,\ a_k = \sum_{e^{k-1}}^{e^k} \frac{\log p_i}{p_i} < \frac{\vartheta(e^k)}{e^{k-1}} < c.$$

Then we obtain from I'' by a simple computation

$$\sum_{k=1}^{n} k\, a_k + O\Big(\sum_{k=1}^{n} a_k \Big) + \sum_{k=1}^{n} a_k (a_1 + a_2 + \ldots + a_{n-k})$$

$$+ O\Big(\sum_{k=1}^{n} a_k a_{n-k+1} \Big) = n^2 + O(n). \quad (3)$$

We obtain (3) from I'' by putting $\dfrac{(\log p_i)^2}{p_i} = \dfrac{\log p_i \log p_i}{p_i}$ $= k \dfrac{\log p_i}{p_i} + O\Big(\dfrac{\log p_i}{p_i}\Big)$, for $e^{k-1} \leqslant p_i < e^k$ and by remarking that if $e^{k-1} \leqslant p_i < e^k$ then in $p_i\, p_j \leqslant e^n$, $e^{n-k} \leqslant p_j < e^{n-k+1}$, also $a_k < c$.

Thus finally we obtain from (3)

$$\sum_{k=1}^{n} a_k (s_{n-k}+k) = n^2+O(n), 0 \leqslant a_k \; (a_k < c \text{ is not needed}$$

any more). Hence from Theorem 2

$$s_n = n + O(1),$$

or in other words II' holds, which proves Theorem 1.

Thus we only have to prove Theorem 2. The difficulty is caused by the sharpness of the error term. First to illustrate our method we prove that

$$s_n = n + O(\log n). \tag{4}$$

In the proof we shall use some of the ideas used in our proof with Selberg[1]. We shall prove (4) in stages. The first step is to prove that

$$s_n = n + o(n). \tag{5}$$

Apply (2) for n and $n+1$ and subtract. Then we obtain

$$(n+1)\, a_{n+1} < 2n + O(n) \text{ (since } a_k \geqslant 0),$$

or

$$a_n \leqslant c. \tag{6}$$

Put

$$A = \overline{\lim} \frac{s_n}{n}, a = \underline{\lim} \frac{s_n}{n}.$$

We obtain from (2) by a very simple argument that $a \leqslant 1, A \geqslant 1$. In fact if $a \leqslant 1$ we have $s_{n-k} + k \geqslant n + o(n)$. Thus from (2)

$$n^2 + O(n) = \sum_{k=1}^{n} a_k (s_{n-k}+k) \geqslant (n + o(n)) \sum_{k=1}^{n} a_k$$

or $A \leqslant 1$, i.e. $a = A = 1$. Hence we can assume $a < 1$, and similarly $A > 1$.

(6) implies $A < \infty$. Choose n so that $s_n = A.n + o(n)$. We clearly can assume $a < A$ (otherwise (5) holds and there is nothing to prove). For $k \leqslant \epsilon n$ (ϵ small but fixed) we have by (6)

$$s_{n-k} + k \geqslant s_n - k\, c + k \geqslant An - \epsilon\, c\, n - \sigma(n) > (a+\delta)\, n, \tag{7}$$

where $\delta = \delta(\epsilon) > 0$ is a fixed number. For $k > \epsilon n$ we have

$$s_{n-k}+k \geqslant a\,(n-k)+k-o(n)=an+k\,(1-a)-o(n)$$
$$> an+\epsilon n\,(1-a)-o(n) > (a+\delta)\,n \qquad (8)$$

Hence from (2), (7) and (8) we have

$$n^2+O\,(n)=\sum_{k=1}^{n} a_k\,(s_{n-k}+k) \geqslant (a+\delta)\,n.\sum_{k=1}^{n} a_k$$
$$=A\,a\,n^2+A\,\delta\,n^2+o(n^2),$$

or $A\,a<1$. By choosing n so that $s_n=an+o(n)$, we obtain similarly $A\,a>1$, an evident contradiction, which proves that $A=a=1$, hence (5) is proved.

Next we prove

$$s_n=n+o(n^\varepsilon). \qquad (9)$$

Put

$$s_n=n+B_n, B_n=o(n), \text{by } (5).$$

We can assume that $\overline{\lim}\ B_n=\infty$ (otherwise there is nothing to prove). Denote

$$\bar{B}_n=\max_{m\leqslant n}\ |B_m|.$$

First we assume that

$$\overline{\lim_{n=\infty}}\ \bar{B}_n/\bar{B}_{n/2}>1+c. \qquad (10)$$

Choose n so that

$$\bar{B}_n/\bar{B}_{n/2}>1+c, \qquad (11)$$

without loss of generality we my assume that $|B_n|=\bar{B}_n$. For if not, we have for some m, $n/2<m\leqslant n$, $|\bar{B}_m|=\bar{B}_n$ and clearly $\bar{B}_m/\bar{B}_{m/2}\geqslant \bar{B}_n/\bar{B}_{n/2}>1+c$. Further we can assume without loss of generality (as will be clear from our proof) that $B_n=\bar{B}_n$. We have by (11) and the definition of $\bar{B}_n$, $\bar{B}_{n/2}$ for every k

$$s_{n-k}+k \geqslant n-k-B_n+k=n-B_n \qquad (12)$$

and for $k>n/2$

$$s_{n-k}+k \geqslant n-k-\bar{B}_{n/2}+k>n-k-B_n/(1+c)+k>n-B_n+c_1\,B_n. \qquad (13)$$

Thus from (2), (12) and (13)

$$n^2+O(n)=\sum_{k=1}^{n} a_k(s_{n-k}+k)>\sum_{k=1}^{n} a_k(n-B_n)+c_1 B_n \sum_{k=n/2}^{n} a_k.$$

Now we have from (5)

$$\sum_{k=1}^{n} a_k=n+B_n,\ \sum_{n/2}^{n} a_k=\frac{n}{2}+o(n)>\frac{n}{4}.$$

Thus finally

$$n^2+O(n)>n^2-B_n^2+c_1 B_n\, n$$

which is false since $\overline{B}_n\,(=B_n)\to\infty$ and $B_n=o(n)$.

Thus we can assume that

$$\lim_{n=\infty} \overline{B}_n/\overline{B}_{n/2}=1. \tag{14}$$

But then we immediately obtain by iterating (14) that $B_n=o(n^\varepsilon)$, which proves (9).

Now we are ready to prove (4). Assume that (4) does not hold. Then we clearly can assume that

$$\overline{\lim}\,(\overline{B}_n-\overline{B}_{n/2})=\infty.$$

Choose n so that $\overline{B}_n-\overline{B}_{n/2}>C$. As before we can assume that $\overline{B}_n=B_n$. As in (12) and (13) we have

$s_{n-k}+k\geqslant n-B_n$ for all k; $s_{n-k}+k>n-B_n+C$ for $k>n/2$.

Thus we have as before

$$n^2+O(n)=\sum_{k=1}^{n} a_k(s_{n-k}+k)>\sum_{k=1}^{n} a_k\,(n-B_n)$$
$$+C\sum_{n/2}^{n} a_k>n^2-B_n^2+\frac{C}{4}\,n,$$

which is false since $B_n=o\,(n^\varepsilon)$ and C can be chosen as large as we please. This contradiction proves (4).

Unfortunately it seems that one cannot get a stronger result by this method.

Now we show that to prove Theorem 2 it will suffice to assume that for sufficiently large k, $a_k<2+\epsilon$, where ϵ is an arbitrarily small but fixed positive number.

Assume then that Theorem 2 is proved in the case $a_k < 2+\epsilon$. Then we handle the general case as follows: Let $t = t(\epsilon)$ be sufficiently large. Put

$$A_k' = \frac{1}{t} \sum_{(k-1)t+1}^{kt} a_u.$$

It is easy to see that $A_k < 2+\epsilon$ for sufficiently large k if $t = t(\epsilon)$ was large enough. To show this, consider

$$S_{kt} - S_{(k-1)t} = 2\,k\,t^2 + t^2 + O(kt)$$

$$\geqslant (k-1)\,t\,(a_{(k-1)t+1} + a_{(k-1)t+2} + \ldots + a_{kt}) = (k-1)\,t^2 A_k,$$

or $A_k < 2+\epsilon$ for $t = t(\epsilon)$. Now we obtain from $0 \leqslant a_u < c$ by a simple calculation

$$\sum_{k=1}^{n} A_k\,(A_1+A_2+\ldots+A_{n-k}+k)$$

$$= t^{\frac{1}{2}} \sum_{u=1}^{tn} a_u\,(a_1+a_2+\ldots+a_{tn-u}+u) + O(n)$$

or

$$\sum_{k=1}^{n} A_k\,(A_1+A_2+\ldots+A_{n-k}+k) = n^2 + O(n).$$

Thus since $A_k < 2+\epsilon$ for sufficiently large k, we obtain $\sum_{k=1}^{n} A_k = n + O(1)$, hence by the definition of the A_k's, $\sum_{k=1}^{n} a_k = n + O(1)$.

Thus it will suffice to prove Theorem 2 in case we have

$$0 \leqslant a_k < 2+\epsilon, \text{ for sufficiently large } k. \qquad (15)$$

Define B_n, $\bar{B}_n$ as before. Without loss of generality we can again assume that $B_n = \bar{B}_n$ and that $\bar{B}_n \to \infty$, from this last assumption we will obtain a contradiction.

We define the u and v numbers as follows: An integer is a v number if

$$s_v < v - B_n + \log B_n.$$

It is a u number if

$$s_u > u + B_n - (\log B_n)^2.$$

First we have to prove some lemmas about the u and v numbers. Let $\delta = \delta(\epsilon)$ be small but fixed, we have

LEMMA I. *Let $y < x$. The number $U(x)$ of u numbers not exceeding x satisfies the following inequality:*

$$U(x) - U(y) < \left(\tfrac{1}{2} + \delta\right)(x - y) + o(x).$$

It clearly suffices to prove the lemma if x is a u number, say u_r. We evidently have

$$u_r^2 + O(u_r) = \sum_{k=1}^{u_r} a_k (s_{u_r - k} + k)$$

$$> (u_r + B_n - (\log B_n)^2)\,(u_r - B_n) + B_n \sum_{k < u_r}{}' a_k, \qquad (16)$$

where the dash indicates that the summation is extended over the k for which $u_r - k$ is a u number (for if $u_r - k$ is a u number $s_{u_r - k} + k > u_r$). From (16) we obtain

$$u_r^2 + O(u_r) \geqslant u_r^2 - B_n^2 - (\log B_n)^2 (u_r - B_n) + B_n \sum_{k < u_r}{}' a_k$$

which implies $[B_n = o(u_r^\varepsilon)$ by (9)$]$

$$\sum{}' a_k = \sum_{j \leqslant r}{}' a_{u_r - u_j} = o(u_r). \qquad (17)$$

But we have from (5)

$$\sum_{k < u_r - y} a_k = u_r - y + o(u_r). \qquad (18)$$

Thus from (17) and (18)

$$\sum_{\substack{k < u_r - y \\ k \neq u_r - u_j}} a_k = u_r - y + o(u_r). \qquad (19)$$

Hence finally from (19) and $a_k < 2 + \epsilon$ we obtain that the sum (19) contains at least $\dfrac{u_r - y}{2 + \epsilon} + o(u_r)$ summands, or

$$U(u_r)-U(y)=U(x)-U(y)<(\tfrac{1}{2}+\delta)\ (x-y)+o(x),$$
which proves Lemma 1.

LEMMA 2.

$$U(n)>(\tfrac{1}{2}-\delta)n.$$

Denote by m the greatest v number not exceeding n. First we show that $m=n+o(n)$. For, if not, then no integer in the interval $[(1-\epsilon)\ n, n]$ would be a v number. But then

$$S_n=n^2+O(n)=\sum_{k=1}^{n} a_k\ (s_{n-k}+k)\geqslant(n-B_n)\sum_{k=1}^{n} a_k+\log\ B_n$$
$$\times\sum_{k\leqslant\epsilon n} a_k=n^2-B_n^2+\epsilon n\log B_n+o(n\log B_n),$$

which is clearly false $(B_n=o(n^\epsilon))$. Thus $m=n+o\ (n)$ is proved. Now we have $S_m=m^2+O(m)$

$$=\sum_{k=1}^{m} a_k\ (s_{m-k}+k)\leqslant(m+B_m)\sum_{k=1}^{m} a_k$$
$$-(\log B_n)^2\sum_{k\leqslant m}^{m}{}'' a_k,$$

where in Σ'' the summation is extended over the k's for which $m-k$ is not a u number. Thus

$$m^2+O(m)\leqslant(m+B_n)\ (m-B_n+\log B_n)-(\log B_n)^2\sum_{k\leqslant m}{}'' a_k$$

which implies

$$\sum_{m-k \text{ not } u} a_k=o(m). \tag{18'}$$

But since $\sum_{k=1}^{m} a_k=m+o(m)$ and $a_k<2+\epsilon$, the sum (18′) can contain at most

$$m-\frac{m}{2+\epsilon}+o(m)$$

summands, which means that $m-k$ is a u number for at least $\frac{m}{2+\epsilon}+o(m)$ values of k, or

$$U(n) = U(m) \geqslant n\ (\tfrac{1}{2}-\delta)$$

which proves the lemma.

Let $z_1, z_2, \ldots$, be any sequence of integers. The number of z's not exceeding n we denote by $C(n)$. The Schnirelmann density of the z's is defined as the lower bound of $C(n)/n$, If the z's are all less than or equal to m, then we take the lower bound only in the interval $1 \leqslant n \leqslant m$ and we then call it the Schnirelmann density up to m.

LEMMA 3. *Let* $1 < u_1 < u_2 < \ldots < u_x \leqslant n$ *satisfy* $x > c_1 n$. *Then there exists a* u_k *so that the sequence* $u_k + 1 - u_i$, $1 \leqslant u_i \leqslant u_k$ *has Schnirelmann density* $\geqslant c$ *up to* u_k.

REMARK. We assumed $u_1 > 1$, since otherwise the lemma is trivial, we can choose $u_k = u_1 = 1$.

Suppose the lemma is false. Then we can clearly cover the interval from 1 to u_x by intervals of the form $(u_x, u_{y_1}+1), (u_{y_1}, u_{y_2}+1), \ldots, (u_{y_k}, 1)$ so that in each of these intervals the number of u's is less than

$$c_1\ (u_{y_i} - u_{y_{i+1}}),\ i = 0, 1, \ldots k\ (x = y_0,\ 0 = y_{k+1}). \quad (19')$$

Adding the inequalities (19) we obtain

$$C(n) = C(u_x) < c_1\, u_x,$$

an evident contradiction, which proves the lemma.

We define now the u' numbers as follows: An integer $t \leqslant n$ is a u' number if there exists a u number u_r so that $|t - u_r| \leqslant \log B_n$. Let $u'_1 < u'_2 < \ldots < u'_x \leqslant n$ be the sequence of consecutive u' numbers. It is easy to see that the u' numbers satisfy Lemmas 1 and 2. For Lemma 2 this is obvious. It is almost obvious for Lemma 1 too. Because of $0 \leqslant a_k < c$ we evidently have

$$s_{u'} > u' + B_n - (\log B_n)^2 - c \log B_n > u' + B_n - 2\ (\log B_n)^2$$

and Lemma 1 can clearly be proved with $s_{u'} > u' + B_n - 2\ (\log B_n)^2$ instead of $s_u > u + B_n - (\log B_n)^2$.

The main advantage of the u' numbers is that they cannot occur in isolation but always occur in bunches of length not less than $2 \log B_n$ $(B_n \to \infty)$. From

$$s'_{u_1} > u'_1 + B_n - 2\,(\log B_n)^2$$

we obtain by (5) that $u'_1 > 1$ (in fact $u'_1 = u'_1\,(n) \to \infty$). Thus Lemma 3 can be applied and we deduce the existence of a u'_y so that the sequence

$$u'_y + 1 - u'_j,\; j \leqslant y$$

has Schnirelmann density $> \frac{1}{2} - \delta$ up to u'_y, also we can assume that u_y is the largest element of its bunch, thus we immediately obtain that the sequence

$$u'_y - u'_j,\; j \leqslant y$$

has Schnirelmann density $\frac{1}{2} - 2\,\delta$ up to u'_y.

We have by (17)

$$\Sigma' a_{u'_y - u'_j} = o(u_y). \tag{20}$$

Now we need the following

LEMMA 4. *Let $z_1 < z_2 < \ldots \leqslant z_r \leqslant N$ be a sequence whose Schnirelmann density up to n, c_2 satisfies $\frac{1}{3} < c_2 < \frac{2}{3}$, and for any $m \leqslant N$ the number $C(m)$ of z's not exceeding m satisfy*[5]

$$C(m) < \tfrac{2}{3}\, m + o(N). \tag{21}$$

Denote further by $y_1, y_2, \ldots y_{N-r}$ the positive integers $\leqslant N$ which are not z's. Then there exists a z, say z_i, so that the sequence $z_j - z_i$ contains more that $N/240\; y$'s.

Denote by $D(N)$ the number of solutions of

$$z_j - U = y_i,\; 0 < U < N.$$

Clearly to each y_j there are exactly $z_j - j$ U's so that

$$D(N) = \sum_{j=1} (z_j - j) \geqslant \sum_{j=1}^{n} (4/r . j - j) - r . o(N)$$

$$\geqslant \frac{1}{6} r^2 - o(N^2) > \frac{1}{54} N^2 - o(N^2)$$

since from (20) $z_j \geqslant \frac{4}{3} j - o(N)$ and $r > \frac{1}{3} N$. Thus there exists an integer U_0 so that the sequence $z_j - U_0$ contains more than $N/60\; y$'s.

Next we prove that every integer $\leqslant N$ is the sum of 4 or less z's. By a well-known theorem of Schnirelmann—Landau[6] the Schnirelmann density up to N of the integers of the form $z_i, z_{j_1}+z_{j_2}$ is $\geqslant 20-c^2>\frac{2}{3}-\frac{1}{9}>\frac{1}{2}$. Thus clearly every number $\leqslant N$ is the sum of 4 or less z's (since if we have more than $m/2$ a's not exceeding m, then m is the sum of two a's). Write

$$U_o = z_{i_1}+z_{i_2}+z_{i_3}+z_{i_4}$$

(where some of the z's may be o). Denote by E_0 the number of y's in z_j-U_o and by $E_r, r=1,2,3,4$ the number of y's in $z_j-z_{i_r}$, $r=1,2,3,4$.

We now prove that

$$E_o \leqslant E_1+E_2+E_3+E_4. \tag{22}$$

Wc prove (22) as follows: The sequence $z_j-z_{i_1}-z_{i2}$ cannot contain more than E_1+E_2 y's for $z_i-z_{i_1}$ contains E_1 y's and from these we obtain not more than E_1 y's of the form $z_j-z_{j_1}-z_{i_2}$. If on the other hand $z_j-z_{i_1}$ is a z, we can get from them at most E_2 y's of the form $z_j-z_{i_1}-z_{i_2}$, (since $z_j-z_{i_2}$ contains E_2 y's). Similarly $z_j-z_{i_1}-z_{i_2}-z_{i_3}$ contains not more than $E_1+E_2+E_3$ z's and $z_j-z_{i1}-z_{i_2}-z_{i_3}-z_{i_4}=z_j-U_0$ contains not more than $E_1+E_2+E_3+E_4$ y's, which proves (22).

Now $E_o>N/60$, hence from (22) we obtain that for some $r\leqslant 4$

$$E_r > N/240$$

which proves Lemma 4.

Consider now the sequence $u'_y-u'_j$. From Lemmas 1, 2 and 3 it follows that Lemma 4 can be applied to it. Thus there exists an u'_k so that the numbers

$$(u'_y-u'_j)-(u'_y-u'_k)=u'_k-u'_j \ (j<k)$$

contain more than $u'_y/240$ positive integers which are not of the form $u'_y-u'_j$. By (17) we get

$$\sum_{j<k} a_{u'_k-u'_j}=o(u'_k)=o(u'_y). \tag{23}$$

Adding (20) and (23) we have

$$\sum_{j<y} a_{u'_y - u'_j} + \sum_{j<k} a_{u'_k - u'_j} = o(u'_y). \tag{24}$$

The sum (24) contains more than

$$(\tfrac{1}{2}-2\delta)\, u'_y + u'_y/240 > \left(\tfrac{1}{2}+\frac{1}{300}\right) u'_y \ (\delta \text{ small}) \tag{25}$$

summands. From (5) we have

$$\sum_{r \leqslant u'_y} a_r = u'_y + o(u'_y). \tag{26}$$

From (25) and (26) we obtain

$$\Sigma\, a_r = u'_y + o(u'_y), \tag{27}$$

where in (27) r runs through the integers not occurring in (24). By (25) we see that the sum (27) contains less than $\left(\frac{1}{2}-\frac{1}{300}\right) u'_y$ summands. Thus clearly the equation

$$a_k > 2 + 1/1000$$

would have infinitely many solutions. This is false for $\epsilon < 1/1000$. This contradiction establishes Theorem 2, and thus the proof of Theorem 1 is complete.

The question can be raised whether a weaker error term than $O(x)$ in I' suffices to deduce that $\lim \vartheta(x)/x = 1$. I can prove that if the error term is $o(x.\log\log x)$, $\vartheta(x)/x \to 1$ can indeed be deduced. I do not know what the best possible result is in this direction and indeed it is possible that $o(x.\log x)$ suffices for the deduction of $\lim \vartheta(x)/x = 1$.

By the methods of the proof of Theorem 2 we can prove the following more general

THEOREM 2′. *Let* $a_k \geqslant 0, f(n) > c, f(n)/n \to 0$ *and*

$$\sum_{k=1}^{n} a_k\,(s_{n-k}+k) = n^2 + O[n\, f(n)].$$

Then

$$s_n = n + O(f(n)).$$

It is possible that in Theorem 2 the condition $a_k \geqslant 0$ can be replaced by $a_k > -c$, clearly some condition for a_k is needed.

FOOTNOTES

1. P. Erdös *Proc. Nat. Acad. Sci.* (U.S.A.) (1949), 374-84. See also A Selberg, *Annals of Math.*, (1949) 305-313.

2. Selberg, *ibid.*

3. The fact that *I* implies *II* was known to Selberg before my investigation started (oral communication).

4. If the p_i are primes I'' was proved by Selberg in an elementary way and used for the elementary proof of Dirichlet's theorem *Annals of Math.*, (1949) 297-304.

5. The method used here is very similar to that used in my paper "On the arithmetical density..." *Acta Arithmetica* Vol. 1.

6. E. Landau, *Über einige neuere Fortschritte der Additiven Zahletheorie*, Cambridge Tract No 35, p. 56. HenryMann's proof of the $\alpha+\beta$ hypothesis in *Annals of Math.* (1942) would enable us to deduce that every integer is the sum of 3 z's.

SUPPLEMENTARY NOTE

BY

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Theorem 2 of the above paper runs as follows:

Let

$$a_k \geqslant 0,\ \sum_{k=1}^{n} a_k\,(s_{n-k}+k) = n^2+O(n)\ (s_m = \sum_{k=1}^{m} a_k). \qquad (1)$$

Then

$$s_n = n+O(1). \qquad (2)$$

I dealt with this result in a lecture at the University of Illinois this summer and several remarks were made by the audience which I propose to discuss here.

Reiner asked whether anything more can be deduced if in (1) we assume that the error term is $o(n)$. If we put $a_1 = 3/2$, $a_{2k+1} = 2$ for $k > 1$, $a_{2k} = 0$, then $\sum_{k-1}^{n} a_k(s_{n-k}+k) = n^2+o(1)$, but $s_n \neq n+o\ (1)$. On the other hand if we assume that there exists an $\epsilon > 0$ so that for $k > k_0$, $a_k < 2-\epsilon$, then indeed $\sum_{k=1}^{n} a_k\,(s_{n-k}+k) = n^2+o\ (n)$ implies $s_n = n+o(1)$. We do not give the proof since it follows that of the original theorem closely.

Hua raised the following questions: What can be deduced if we assume that $a_k \geqslant 0$ and $\sum_{k=1}^{n} k\,a_k = \frac{1}{2}n^2+O(n)$, also $a_k \geqslant 0$, and $\sum_{k=1}^{n} a_k\ (s_{n-k}+k) = \frac{1}{2}\,n^2+O(n)$?

Here I prove

Theorem 1. *Let $a_k \geqslant 0$ and $\sum_{k=1}^{n} k.a_k = \frac{1}{2}n^2+O(n)$, then*

$$s_n = n+O(\log n). \qquad (3)$$

and (3) is best possible.

( )

To prove (3) put $s_n = n + A_n$. Denote $\max_{m \leqslant n} |A_m| = \bar{A}_n$. We can assume that $\bar{A}_n \to \infty$ (for otherwise (3) holds and there is nothing to prove). Since $\bar{A}_n \to \infty$ we can choose arbitrarily large values of n so that $\bar{A}_n = |A_n|$, and in fact it will be clear from the proof that without loss of generality we can assume $\bar{A}_n = A$. We have

$$\sum_{k=1}^{n} k a_k = n s_n - \sum_{k=1}^{n-1} s_k = n\ (n+\bar{A}_n)$$

$$- \sum_{k=1}^{n-1} (k+A_k) \geqslant \tfrac{1}{2} n^2 + O(n) + \frac{n}{2}(\bar{A}_n - \bar{A}_{n/2}) \quad (4)$$

(if $n/2 < k \leqslant n$ we replace A_k by $\bar{A}_n$, if $k \leqslant n/2$ we replace A_k by $\bar{A}_{n/2}$). If (3) does not hold then clearly $\overline{\lim}\ \bar{A}_{n/\log n} = \infty$, or for every C there exist infinitely many n so that $\bar{A}_n - \bar{A}_{n/2} > C$. But then from (4)

$$\sum_{k=1}^{n} k.a_k > \tfrac{1}{2} n^2 + \frac{C}{2} n + O(n),$$

which contradicts the assumptions of Theorem 1 (since C can be chosen arbitrarily large), which proves (3).

The fact that (3) cannot be improved is immediately clear by putting $a_k = 1 + 1/k$.

Theorem 2. *Let* $a_k \geqslant 0$, $\sum_{k=1}^{n} a_k s_{n-k} = \frac{1}{2} n^2 + O(n)$. *Then*

$$s_n = n + o(n). \quad (5)$$

The error term cannot be $o(n^{1/2})$.

To prove this it suffices to assume that $a_k \geqslant 0$ and $\sum_{k=1}^{n} a_k s_{n-k} = \frac{1}{2} n^2 + o(n^2)$. Put $F(x) = \sum_{k=1}^{\infty} a_k x^k$, $F(x)^2 = \sum_{k=1}^{\infty} b_k x^k$. Clearly

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} a_k s_{n-k} = \tfrac{1}{2}n^2+o(n^2).$$

Thus

$$\lim_{x=1} (1-x)^2 F(x)^2 = 1 \text{ or } \lim_{x=1} (1-x) F(x) = 1.$$

Hence by the well-known Tauberian theorem of Hardy and Littlewood $s_n = n+o(n)$.

By putting $a\,(n\,!)^2 = n\,!, a_m = 0$ if $(n\,!)^2 < m \leqslant (n\,!)^2+n\,!$, $a_m = 1$ otherwise, we immediately obtain that the error term in (5) cannot be $o(n^{1/2})$.

Let $f(x)$ be an increasing function satisfying $f(x) \leqslant x$, $f'(x) \leqslant 1$. $f^{-1}(x)$ is defined by $f[f^{-1}(x)] = x$. Then we have

THEOREM 3. *Let* $a_k \geqslant 0$ *and*

$$S_n = \sum_{k=1}^{n} a_k \left[s_{f^{-1}[f(n)-f(k)]}+f(k)\right] = f(n)^2+O(f(n)). \quad (6)$$

Then

$$s_n = f(n)+O(1). \quad (7)$$

REMARK: If $f(x) = x$ we obtain our original theorem that (1) implies (2), also $f(x) = x^\alpha$, $0 < \alpha \leqslant 1$, $f(x) = \log x$ satisfy the conditions of Theorem 3.

PROOF OF THEOREM 3. Denote $[f(n)] = N$,

(i.e. $f^{-1}(N) = n+\delta, |\delta| < 1$) $\sum_{r<f(n)<r+1} a_k = A_r$. We have

from (6)

$$S_{f^{-1}(N+1)} - S_{f^{-1}(N)} = O(N) \geqslant N A_N \text{ or } A_N < c.$$

Thus from (6) by a simple computation, we have

$$\sum_{r=1}^{N} A_r(A_1+\ldots+A_{N-k_r}+k_r) = N^2+O(N)$$

which by our theorem clearly implies (7).