ON THE NUMBER OF TERMS OF THE SQUARE OF A POLYNOMIAL

BY

P. ERDÖS

Syracuse University

Let \( f_k(x) = a_0 + a_1x^n + \ldots + a_{k-1}x^{n-k+1}, a_i \neq 0, \) for \( 0 < i < k - 1, a_i \) real, be a polynomial of \( k \) terms. Denote by \( Q(f_k(x)) \) the number of terms of \( f_k(x)^2 \). Put

\[
Q(k) = \min Q(f_k(x)),
\]

where \( f_k(x) \) runs through all polynomials having \( k \) non-vanishing terms and real coefficients.

RÉDEI\(^1\) raised the problem whether \( Q(k) < k \) is possible. RéNYI, KALMÁR and RÉDEI\(^1\) proved in fact that \( \liminf_{k \to \infty} Q(k)/k = 0 \), also that \( Q(29) < 28 \). RéNYI\(^1\) further proved that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{Q(k)}{k} = 0.
\]

He also conjectured that

\[
\lim_{k \to \infty} \frac{Q(k)}{k} = 0. \tag{1}
\]

In this note we are going to prove (1), by a slight modification of the method used by RéNYI. In fact we shall prove the following

**Theorem.** There exist constants \( 0 < c_2 \) and \( 0 < c_1 < 1 \), so that

\[
Q(k) < c_2 k^{1-c_1}. \tag{2}
\]

First we state two lemmas, both of which are contained in Rényi's paper.

**Lemma I.** $Q(29) < 28$.

**Lemma II.** $Q(a.b) < Q(a).Q(b)$. (Lemma II in fact is almost obvious).

From lemmas I and II we immediately obtain that

$$Q(29^t) < 28^t,$$  \hspace{1cm} (3)

or (2) is satisfied for integers of the form $29^t$. Assume now $l > 2$ and $29^t < k < 29^{t+1}$.

Put

$$t = \left\lfloor \frac{l}{2} \right\rfloor, \quad r + 1 = 29^t. \hspace{1cm} (4)$$

Let $h(x) = a_1 + a_2 x + \ldots + a_r x^r, a_i \neq 0$, be the polynomial for which $h(x)^2$ has $Q(29^t) < 28^t$ terms. Consider now

$$F(x) = h(x)g(x), \quad g(x) = b_0 + b_1 x^{s+1} + \ldots + b_s x^{snr} + \ldots + b_r x^{snr}, \quad b_i \neq 0$$

where the $b$'s and $s$ will be determined later. Let us compute the number of terms of $F(x)$. Clearly $F(x)$ has exactly $(r - 1)(s + 1)(u + 1)$ terms $d_u x^u$ where $u \equiv 0 \pmod{n}$, further the constant term and the coefficient of $x^{(s+1)nr}$ can not be 0. By suitable choice of the $b$'s we can clearly arbitrarily prescribe whether the coefficient of $x^{snr}$, $1 < u < s$ is 0 or not (we only have to solve equations of the first degree). Thus $g(x)$ can be so chosen that $F(x)$ should have $2 + (r - 1)(s + 1) + A$ terms where $0 < A < s$ is arbitrary. Put

$$s + 1 = \left\lfloor \frac{k}{r-1} \right\rfloor. \hspace{1cm} (5)$$

Clearly by (4) $s \geq r - 1$. Thus

$$2 + (r - 1)(s + 1) < k < 2 + (r - 1)(s + 1) + s.$$

Thus by what has been said before we can determine $g(x)$ so that $F(x) = g(x)h(x)$ has $k$ terms. But then by lemma II $F(x)^2$ has not more than

$$(2s + 2).28^t < c_2 k^{1-c_1} \quad (by (4) \text{ and } (5)).$$
terms \((g(x)^2\) has \(\leq 2s + 2\) terms). Thus our Theorem is proved.

It would be interesting to determine the order of \(Q(k)\) more accurately. Rényi \(^2\) conjectured that \(\lim_{k \to \infty} Q(k) = \infty\), but neither of us could prove this as yet.

One final remark: Since Rényi proves that \(Q(29) \leq 28\) for polynomials having rational coefficients, our proof gives \(Q(k) \leq c_2 k^{1-c_1}\) for polynomials with rational coefficients. Rényi \(^1\) asks whether \(Q(k)\) is the same if the coefficients are rational, real, or complex.

\(^{1}\) Ingel (14.7.48).

\(^{2}\) Oral communication.
SOME THEOREMS ON THE ROOTS OF POLYNOMIALS.

BY

N. G. DE BRUIJN

(Mathematisch Instituut der Technische Hogeschool, Delft).

Our first theorem concerns a composition theorem for polynomials whose roots lie in certain sectors. It is an extension of a result of L. Weissner ¹) and can be proved by the same method. Here we state it, and we give a short independent proof, since it will be applied below.

The term sector is used here in the sense of an open point set in the complex plane bounded by two ²) half lines starting from the origin. If $S_\alpha$ and $S_\beta$ are sectors with aperture $\alpha$ and $\beta$, respectively, and if $\alpha + \beta \leq 2\pi$, then the product $S_\alpha S_\beta$, consisting of all points $w_1 w_2$ ($w_1 \in S_\alpha$, $w_2 \in S_\beta$) is also a sector, with aperture $\alpha + \beta$. The sector consisting of all points $-w$ ($weS$) is denoted by $-S$.

Theorem 1. Put

\[ A(z) = \sum_{n=0}^{M} a_n z^n \]  \hspace{1cm} (a_M \neq 0) \]

\[ B(z) = \sum_{n=0}^{N} b_n z^n \]  \hspace{1cm} (b_N \neq 0) \]

\[ g(z) = \sum_{n=0}^{K} n! a_n b_n z^n \]  \hspace{1cm} (K = \text{Min} (M, N)) \]

Suppose that the roots of $A(z)$ all lie in the sector $S_\alpha$ ($\alpha \leq \pi$) and those of $B(z)$ in $S_\beta$ ($\beta \leq \pi$). Then the roots of $g(z)$ all lie in the sector $S = -S_\alpha S_\beta$.


²) A single half line in case of aperture $2\pi$. 