ON THE STRONG LAW OF LARGE NUMBERS

BY

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In the present note \( f(x) \), \( -\infty < x < \infty \), will denote a function satisfying the following conditions: (1) \( f(x+1) = f(x) \), (2) \( \int_0^1 f(x) = 0 \), \( \int_0^1 f(x)^2 = 1 \). By \( n_k < n_{k+1} < \cdots \) we shall denote an arbitrary sequence satisfying \( n_{k+1}/n_k > c > 1 \), and by \( S_n(f) \) the \( n \)th partial sum of the Fourier series of \( f(x) \).

In a recent paper Kac, Salem, and Zygmund(1) prove (among others) that if for some \( \epsilon > 0 \)

\[
\int_0^1 (f(x) - \phi_n(f))^2 = O\left(\frac{1}{(\log n)^{\epsilon}}\right),
\]
then for almost all \( x \)

\[
\lim_{N \to \infty} \frac{1}{N} \left( \sum_{k=1}^N f(n_kx) \right) = 0,
\]
or roughly speaking the strong law of large numbers holds for \( f(n_kx) \) (in fact the authors prove that \( \sum f(n_kx)/k \) converges almost everywhere).

The question was raised whether (2) holds for any \( f(x) \). This was known for the case \( n_k = 2^k \). In the present paper it is shown that this is not the case. In fact we prove the following theorem.

**Theorem 1.** There exists an \( f(x) \) and a sequence \( n_k \) so that for almost all \( x \)

\[
\limsup_{N \to \infty} \frac{1}{N} \left( \sum_{k=1}^N f(n_kx) \right) = \infty.
\]

Further we prove the following sharpening of the result of Kac-Salem-Zygmund:

**Theorem 2.** Assume that for some \( \epsilon > 0 \)

\[
\int_0^1 (f(x) - \phi_n(f))^2 = O\left(\frac{1}{(\log \log n)^{\epsilon+\epsilon}}\right),
\]
then (2) holds.

By a slight modification of the construction of the \( f(x) \) of Theorem 1 it is easy to construct an \( f(x) \) and a sequence \( n_k \) for which (3) holds and for which

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(2) This result is due to Raikov. See F. Riesz, Comment. Math. Helv. vol. (17) (1944) p. 223.
There is clearly a gap between (4) and (5). It seems probable that, in Theorem 2, (4) can be replaced by $1/(\log \log \log n)^2$, but much sharper methods would be needed than used here.

The following problem also seems of some interest: By an easy modification in the construction of the $f(x)$ of Theorem 1 we can show the existence of an $f(x)$ and a sequence $n_k$ so that for almost all $x$

$$\limsup_{N \to \infty} \frac{1}{N(\log \log N)^{1/2}} \left( \sum_{l=1}^{N} f(n_k x) \right) = \infty.$$  

(6)

On the other hand we can show that for almost all $x$

$$\lim_{N \to \infty} \frac{1}{N(\log \log N)^{1/2}} \left( \sum_{l=1}^{N} f(n_k x) \right) = 0.$$  

(7)

Again there is a gap between (6) and (7). (6) seems to give the right order of magnitude, but I can not prove this.

One final remark. The $f(x)$ of Theorem 1 is unbounded. The possibility that (2) holds for all bounded functions $f(x)$ remains open.

**Proof of Theorem 1.** Let $u_k, v_k, A_k$ tend to infinity sufficiently fast (their growth will be specified later). $r_m(x)$ denotes the $m$th Rademacher function. Put

$$f(x) = \sum_{k=1}^{w} \sum_{m=n_k+1}^{v_k} \frac{v_m(x)}{A_k(v_k - u_k)^{1/2}} \sum_{k=1}^{w} \frac{1}{A_k} = 1.$$  

(8)

Clearly the series for $f(x)$ converges almost everywhere and $\int f(x) = 0$, $\int f(x)^2 = 1$. Now we define the $n_k$. Put $j_k = \lfloor \log N \rfloor$. Denote by $I^{(k)}_t$ the interval $((2t-1)v_k, (2t-1)v_k + l^{(k)}_t)$, $t = 1, 2, \ldots, j_k$,

where $l^{(k)}_1 = 2l^{(k)}_1$ and $l^{(k)}_t$ is very large compared to $v_k-1, A_k-1, l^{(k-1)}_t$, and will be specified later. If $v_k > l^{(k)}_{j_k}$ then the $I^{(k)}_t$ don’t overlap. The $n_k$ are the integers of the form $2^m$ where $m \subseteq I^{(k)}_t, k = 1, 2, \ldots ; t = 1, 2, \ldots, j_k$. Order the $l$’s according to their size. Clearly each $l$ is greater than the sum of all previous $l$’s. Thus a simple argument shows that to prove (3) it will be sufficient to show that for every fixed $\epsilon$ and almost all $x$

$$\limsup_{l^{(k)}_t} \frac{1}{l^{(k)}_t} \left( \sum_{m \subseteq I^{(k)}_t} f(2^m x) \right) > \epsilon,$$  

(9) \hspace{1cm} k = 1, 2, \ldots ; t = 1, 2, \ldots, j_k.$$

(9) Instead of $r_m(x)$ I originally used $\cos 2^m x$. The advantage of using Rademacher functions was pointed out to me by Kac.
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(3) is a consequence of (9). Hence it will suffice to show that for every \( \epsilon \) and sufficiently large \( k \) the measure of the set in \( x \) satisfying at least one of the inequalities

\[
\frac{1}{n^t} \left( \sum_{m \in I_t(k)} f(2^m x) \right) > c, \quad t = 1, 2, \ldots, j_n
\]
is greater than \( 1 - \epsilon \).

Put

\[
f(x) = f_1(x) + f_2(x) + f_3(x)
\]

where

\[
f_1(x) = \sum_{m=1}^{\infty} \sum_{m=u_k+1}^{v_k} \frac{r_m(x)}{(A_1(v_k - u_k))^{1/2}}, \quad f_2(x) = \sum_{m=1}^{v_k} \frac{r_m(x)}{(A_2(u_k - v_k))^{1/2}}, \quad f_3(x) = \sum_{m=1}^{u_k} \frac{r_m(x)}{(A_3(u_k - v_k))^{1/2}}.
\]

A simple calculation shows that

\[
\sum_{m \in I_t(k)} f_2(2^m x) = \frac{l_t^{(k)}}{(A_2(u_k - v_k))^{1/2}} \sum_{m} r_m(x) + \sum_{1} + \sum_{2}
\]

where \( m \) runs in the interval

\[
(u_k + (2t - 1)v_k + l_t^{(k)}, 2tv_k)
\]

and

\[
\sum_{1} = \sum_{a=1}^{l_t^{(k)}} \frac{l_t^{(k)} - a}{(A_2(v_k - u_k))^{1/2}} r_y - a(x), \quad y = u_k + (2t - 1)v_k + l_t^{(k)},
\]

\[
\sum_{2} = \sum_{a=1}^{l_t^{(k)}} \frac{l_t^{(k)} - a}{(A_2(v_k - u_k))^{1/2}} r_a(x).
\]

Now \( \sum r_m(x) \) is the sum of

\[
v_k - u_k - l_t^{(k)} > v_k/2
\]

Rademacher functions (we choose \( v_k > 2(u_k + l_t^{(k)}) \)). It is well known(1) that

\[(1) \text{ See, for example, P. Erdős, Ann. of Math. vol. 43 (1942) p. 420, formula (0.7). Incidentally the formula in question should read }\]

\[
\Pr(A_n(x)) \leq e^{-c_1 n^{1/2}}, \quad \Pr(A_n(x)) < c_2(x/n) e^{-c_2 n}.
\]
the measure of the set in \( x \) for which
\[
\sum r_{m}(x) > 4c(A_k)^{1/2}(v_k)^{1/2}
\]
is greater than
\[
e_{1, A_k} e^{-32c_{1}A_k} > e^{-A_k^{4}}
\]
for sufficiently large \( A_k \). Thus the measure of the set in \( x \) for which
\[
(12) \quad \sum_{m} \frac{t_{i}^{(k)}}{(A_k(v_k - u_k))^{1/2}} \sum r_{m}(x) > 4c_{l_{i}}^{(k)}
\]
is greater than \( e^{-A_k^{2}} \). Clearly for all \( x \)
\[
(13) \quad |\sum_{s} + \sum_{s}| < \frac{2(t_{i}^{(k)})^{2}}{(A_k(v_k - u_k))^{1/2}} < \frac{4(t_{i}^{(k)})^{2}}{(v_k)^{1/2}} < 1
\]
if we choose \( v_k > 16(l_{i}^{(k)})^{4} \). Thus finally from (11), (12), and (13) the measure of the set in \( x \) for which
\[
(14) \quad \sum_{m \in I_{i}^{(k)}} f_{2}(2^{m} x) > 4c_{l_{i}}^{(k)} - 1 > 3c_{l_{i}}^{(k)}
\]
is greater than \( e^{-A_k^{2}} \).

If \( v_k > 2(l_{i}^{(k)}) \) for all \( t \), then the functions
\[
\sum_{m \in I_{i}^{(k)}} f_{2}(2^{m} x), \quad t = 1, 2, \ldots, j_k,
\]
are independent (since the same \( r_{m}(x) \) does not appear in two different sums). Thus the measure of the set in \( x \) for which one of the \( j_k \) inequalities
\[
(15) \quad \sum_{m \in I_{i}^{(k)}} f_{2}(2^{m} x) > 3c_{l_{i}}^{(k)}, \quad t = 1, 2, \ldots, j_k,
\]
holds, is greater than
\[
(16) \quad 1 - (1 - 1/y)^{s} > 1 - e/2 (y = e^{k}, s = e^{A_k^{2}}).
\]
Further if \( l_{i}^{(k)} > v_{k-1} \)
\[
\int_{0}^{1} \left( \sum_{m \leq l_{i}^{(k)}} f_{1}(2^{m} x) \right)^{2} < 2 v_{k-1}(l_{i}^{(k)}) + v_{k-1} < 2 v_{k-1}^{2} \]
since only the \( r_{m} \)'s with \( m \leq l_{i}^{(k)} + v_{k-1} \) occur and the coefficients of all of them are not greater than \( v_{k-1} \). Thus from Tchebychev's inequality we obtain that the measure of the set in \( x \) for which one of the \( j_k \) inequalities
\[
(17) \quad \sum_{m \in I_{i}^{(k)}} f_{1}(2^{m} x) > c_{l_{i}}^{(k)}, \quad t = 1, 2, \ldots, j_k,
\]
holds is less than
\[
\sum_{k=1}^{t_k} \frac{2^{\mu(k-1)}}{c_{l_k}^{(2)}} < \frac{4^{\nu(k-1)}}{c_{l_k}^{(3)}} < \frac{\epsilon}{4}, \quad \text{for } t_k > 16v_k^{-1}/\alpha.
\]

Finally we have by a simple computation
\[
\int_0^1 \left( \sum_{n \in T_k(x)} f_3(2^m x) \right)^2 < 4(\tau_k^{(3)})^2 \sum_{t \geq k} \frac{1}{A_t} < 1
\]
if \( A_k, \ldots \) are sufficiently large. Thus the measure of the set in \( x \) for which one of the inequalities
\[
\sum_{n \in T_k(x)} f_3(2^m x) > c_{l_k}^{(k)}, \quad t = 1, 2, \ldots, j_3,
\]
holds is less than
\[
\sum_{k=1}^{t_k} \frac{1}{(c_{l_k}^{(k)})^2} < \frac{\epsilon}{4}.
\]
Thus finally from (15), (16), (17), (18), (19), and (20) we obtain (10) and this completes the proof of Theorem 1.

Sketch of the Proof of Theorem 2. Put \( j = r \), then \( n_j/n_i > \sigma \). Denote by \( a_1, b_1, a_2, b_2, \ldots \) the Fourier coefficients of \( f(x) \). By (4) we evidently have
\[
\int_0^1 f(n_1 x) f(n_j x) = \sum_{a_1, b_1 = n_1} (a_1 a_2 + b_1 b_2) \leq \left( \sum_{a_1} a_1^2 \sum_{b_2} b_2^2 \right)^{1/2} + \left( \sum_{b_1} b_1^2 \sum_{a_2} a_2^2 \right)^{1/2} < \frac{c_1}{(\log r)^{1+\epsilon/2}}.
\]

Hence
\[
\int_0^1 \left( \sum_{n \in M_A(x)} f(n x) \right)^2 = o\left( \frac{N^2}{(\log N)^{1+\epsilon/2}} \right),
\]
or the measure of the set \( M_A(x, N, A) \) in \( x \) for which
\[
\left| \sum_{n \in M_A(x)} f(n_k x) \right| > A \cdot N
\]
is less than
\[
c/A^2(\log N)^{1+\epsilon/2}.
\]
Consider the sets
\[
M(1, 2^n, \delta); M(2^n, 2^{n-1}, 2\delta/2^n);
M(2^n, 2^{n-2}, 4\delta/3^n), M(2^n + 2^{n-1}, 2^n, 4\delta/3^n); \ldots .
\]
There are $2^{k-1}$ sets of order $k$, that is, sets of the form
\[(23) \quad M(2^n + 2u 2^{n-k}, 2^{n-k}, \delta 2^k/(k+1)^2), 0 \leq u < 2^{k-1}.
\]
From (21) it follows that the measure of any set of order $k$ does not exceed
\[c(k+1)^{4/\delta^2}2^{k(n-k)^{1+1/2}}.
\]
Thus the measure of all the sets in (23) is less than $c(k+1)^{4/\delta^2}2^{k(n-k)^{1+1/2}}$, and the measure of all the sets $M_n$ in (22) does not exceed
\[
\sum_{k=0}^{n} \frac{c(k+1)^{4/\delta^2}2^{k(n-k)^{1+1/2}}}{\delta^2 n^{1+1/2}} < c1.
\]
Thus
\[(24) \quad \sum_{n=1}^{\infty} M_n < \infty.
\]
But if $x$ does not belong to any of the sets (22) we have by a simple argument for all $2^n \leq m < 2^{n+1}$ (every $m$ is the sum of powers of 2)
\[(25) \quad \left| \sum_{k=1}^{m} f(n,x) \right| < \delta 2^n + \frac{\delta 2^n}{2^2} + \frac{\delta 2^n}{3^2} + \cdots + \frac{\delta 2^n}{k^2} + \cdots < 2\delta 2^n \leq 2\delta m.
\]
(24) and (25) clearly prove theorem 2(\textsuperscript{\textdegree}).

\textsuperscript{\textdegree} The method used here is due to Hobson-Plancherel-Rademacher-Menchof. (See, for example, Rademacher, Math. Ann. vol. 87 (1922) p. 117–121.)