SOME PROBLEMS AND RESULTS ON CONSECUTIVE PRIMES

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§ 1. Let $p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4, \dots p_n, \dots$ denote the sequence of primes. In what follows we shall be concerned with some problems regarding difference and quotient of consecutive primes. We put

$$(1) \qquad d_n = p_n - p_{n-1},\ q_n = \frac{p_n}{p_{n-1}}, \qquad n = 2, 3, \dots .$$

There are many unsolved problems regarding the sequences d_n and q_n, and the subject is full of peculiar difficulties. For instance it has been conjectured by many authors that $d_n = 2$ for infinitively many values of n, i.e. that the sequence of "twin primes":

3,5 5,7 11,13 17,19 29,31 41,43 ...

is infinite. Neither this, nor the more feeble conjecture that d_n does not tend to infinity, has been proved up to now. Moreover at present we are unable to prove even that $\lim_{n\to\infty} \frac{d_n}{\log n} = 0$.

In this direction it has been proved by P. Erdös [1]) that

$$(2) \qquad \lim_{n\to\infty} \frac{d_n}{\log n} \leq C < 1 \qquad *)$$

Recently R. A. Rankin [2]) proved, that for C we can choose $\frac{59}{61}$, and according to an oral communication of A. Selberg he can improve this result by choosing for C the value $\frac{15}{16}$. This is all we know about the small values of d_n. In the opposite direction it can be proved by the method of Viggo Brun that twin primes and more generally pairs of consecutive primes the difference of which is equal to a fixed (even) integer $2k$, are rather "few". More precisely, we have the following result [3]): The number of solutions of $d_n = k$, $p_n \leq x$ does not exceed

$$(3) \qquad \frac{c_1 x}{\log^2 x} \prod_{p/k} \left(1 + \frac{1}{p}\right) \qquad **)$$

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*) From the prime number theorem it follows only $\lim_{n\to\infty} \frac{d_n}{\log n} \leq 1$.

**) We denote by $c_1, c_2, \dots$ positive constants, and by c a positive constant which is not always the same.

It follows from (3) that the sequence d_n is not bounded. As a matter of fact, this follows also from TCHEBYSHEFF's estimation [4])

$$\vartheta(x) = \sum_{p_n \leq x} \log p_n < c_1 x,$$

and also from the trivial remark that no one of the consecutive numbers $n! + 2$, $n! + 3$, $n! + 4, \ldots n! + n$ is prime, and thus there are arbitrarily large "gaps" in the sequence of primes. From TCHEBYSHEFF's estimation, mentioned above, it follows also that $\overline{\lim\limits_{n\to\infty}} \dfrac{d_n}{\log p_n} \geqq \dfrac{1}{c_1}$. According to the primes number theorem we can choose $c_1 = 1 + \varepsilon$ for any $\varepsilon > 0$, if x is sufficiently large and thus we obtain $\overline{\lim\limits_{n\to\infty}} \dfrac{d_n}{\log p_n} \geqq 1$. In his paper [5]) where previous results are quoted, P. ERDÖS proved, that

$$\overline{\lim_{n\to\infty}} \frac{d_n}{\dfrac{\log p_n \,.\, \log\log p_n}{(\log\log\log p_n)^2}} > 0. \tag{4}$$

This has been still improved by RANKIN [6]) by adding in the numerator of the denominator in (4) the factor loglogloglog p_n.

§ 2. If the conjecture regarding twin primes would be proved, it would follow that the sequence d_n oscillates between 2 and arbitrary large values, thus it would be neither monotonically increasing nor decreasing, from some point onwards. This result has been proved without any hypothesis, by P. ERDÖS and P. TURÁN [7]). They proved also that the sequence q_n is also neither monotonically increasing nor decreasing. These results can be expressed also by saying that the sequences p_n and $\log p_n$ are neither convex nor concave from some point onwards. A generalization of this result, regarding the sequence $\log p_n$, has been given recently by A. RÉNYI [8]). His result formulated in a geometrical terminology, runs as follows: Let us consider a finite polygonal line situated in the complex z-plane and having the consecutive vertices $z_1, z_2, \ldots z_n$. The total curvature of this polygonal line is defined by

$$G = \sum_{k=2}^{n-1} \left| \arg \frac{z_{k+1} - z_k}{z_k - z_{k-1}} \right| \tag{5}$$

where $\arg Z$ denotes the argument of the complex number Z, i.e. if $Z = r \,.\, e^{i\varphi}$, $r > 0$, $-\pi < \varphi \leqq \pi$, we have $\arg Z = \varphi$. Now let π_N denote the polygonal line with the vertices $z_n = n + i \,.\, \log p_n$, $p_{n+1} \leqq N$ and let G_N denote its total curvature. It is evident, that if the sequence $\log p_n$ would be convex or concave from some point onwards, G_N would be bounded. But RÉNYI proved that G_N tends to infinity, as a matter of fact he proved $G_N > c. \log\log\log N$, thus giving a new proof of the theorem of ERDÖS and TURÁN mentioned above. In waht follows we shall prove a refinement of this result, giving the exact order of magnitude of G_N:

Theorem 1. If G_N denotes the total curvature, defined by (5), of the polygonal line π_N having the vertices $z_n = n + i. \log p_n$, $p_{n+1} \leqq N$, we have

$$c_2 . \log N < G_N < c_3 . \log N. \tag{6}$$

It seems probable that $\lim\limits_{N+\infty} \frac{G_N}{\log N}$ exists, but we are at present not able to prove this. The proof of Theorem 1 will be completely elementary. Besides the estimations of TCHEBYSHEFF

$$c_4 \frac{x}{\log x} < \pi(x) < c_5 \frac{x}{\log x}, \quad *) \tag{7}$$

and $p_{n+1} < 2p_n$, we shall use only the method of BRUN.

§ 3. *Proof of Theorem* 1. Clearly we have

$$G_N = \sum_{p_{n+1} \leq N} |\operatorname{arc\,tg} \log q_{n+1} - \operatorname{arc\,tg} \log q_n|. \tag{8}$$

Using $\operatorname{arc\,tg} a - \operatorname{arc\,tg} b = \operatorname{arc\,tg} \frac{a-b}{1+ab}$ this gives

$$G_N = \sum_{p_{n+1} \leq N} \left| \operatorname{arc\,tg} \frac{\log \frac{q_{n+1}}{q_n}}{1 + \log q_{n+1} . \log q_n} \right|. \tag{9}$$

As according to (7) the sequence q_n is bounded, we obtain from (9)

$$C_6 L_N < G_N < L_N \tag{10}$$

where

$$L_N = \sum_{p_{n+1} \leq N} \left| \log \frac{q_{n+1}}{q_n} \right|. \tag{11}$$

As regards the upper estimation, it can be finished at once: we have

$$L_N < 2 \sum_{p_n \leq N} |\log q_n| = 2 \sum_{p_n \leq N} \left| \log\left(1 + \frac{du}{p_{n-1}}\right)\right| \leqq 4 \sum_{p_n \leq N} \frac{du}{p_n}. \tag{12}$$

Now, we obtain

$$\sum_{p_n \leq N} \frac{dn}{p_n} \leqq \sum_{2^k \leq N} \frac{1}{2^k} \Big(\sum_{2^k < p_n < 2^{k+1}} d_n \Big) \leqq \sum_{2^k \leq N} 1 \leqq \frac{\log N}{\log 2}. \tag{13}$$

*) $\pi(x)$ denotes the number of primes which are $\leqq x$.

Thus the upper estimation of (6) is proved. As regards the lower estimation it is somewhat more intricate. We proceed as follows: Let us neglect the terms $\left|\log \frac{q_{n+1}}{q_n}\right|$ of L_N for those indices n for which either d_n or d_{n+1} exceeds $\lambda \log p_n$,*) and denote the remaining sum by L_N'; evidently $G_N > c\, L_N > c\, L_N'$. As according to (7) $\left|\frac{q_{n+1}}{q_n} - 1\right| < 1$, and we have for $|x| < 1$ $|\log(1+x)| > c_7 |x|$, further using the identity

$$(14) \qquad \frac{q_{n+1}}{q_n} = 1 + \frac{d_{n+1} - d_n}{p_n^2} - \frac{d_{n+1}\, d_n}{p_n^2}$$

we obtain, by putting

$$(15) \qquad D_N = \sum_{p_{nH} \leq N}' \frac{|d_{n+1} - d_n|}{p_n}, \quad R_N = \sum_{p_{n+1} \leq N}' \frac{d_{n+1}\, d_n}{p_n^2}$$

(where Σ' denotes that the summation is extended only over such indices n for which both d_n and d_{n+1} are $< \lambda \log p_n$) we obtain

$$(16) \qquad L_N > c_7 (D_N - R_N)$$

Now we have

$$(17) \qquad R_N < \lambda^2 \sum_{p_{n+1} \leq N} \frac{\log^2 p_n}{p_n^2} < \lambda^2 \sum_{k=2}^{\infty} \frac{\log^2 k}{k^2}.$$

Thus if we prove $D_N \gtrsim c_8 \log N$, $G_N > c_2 \log N$ follows. To estimate D_N we need the following lemma.

Lemma 1. The number of solutions of $d_n = a$, $d_{n+1} = b$, $p_n < N$, does not exceed

$$(18) \qquad \frac{c_9 N}{\log^3 N} \prod_{p/a}\left(1 + \frac{1}{p}\right) \prod_{p/a+b}\left(1 + \frac{1}{p}\right).$$

In other words, the number of such primes $p \leq N$ for which both $p + a$ and $p + a + b$ are also primes, does not exceed (18). This lemma can be proved easily by the method of VIGGO BRUN (l.c.) by applying a „triple sieve". We need further

Lemma 2. Let us denote

$$(19) \qquad W(m) = \prod_{p/m}\left(1 + \frac{1}{p}\right).$$

*) The value of the constant λ shall be determined later on.

We have

$$(20) \qquad \sum_{A \leq m \leq A+B} W(m) \leqq \frac{\pi^2}{6} B + \log(A+B) + 1.$$

Proof of Lemma 2. Clearly

$$\sum_{A \leq m \leq A+B} W(m) = \sum_{d \leq A+B} \frac{|\mu(d)|}{d} \cdot \sum_{A \leq dr \leq A+B} 1$$

and thus

$$\sum_{A \leq m \leq A+B} W(m) \leqq \sum_{d \leq A+B} \frac{1}{d}\left(\frac{B}{d} + 1\right) < \frac{B\pi^2}{6} + \log(A+B) + 1.$$

Now we prove, using Lemmas 1 and 2, the following

Lemma 3. Let ε, λ and M denote positive constants, and let k denote a positive integer, $k \geqq k_0(M, \varepsilon, \lambda)$. The number of indices n, for which $d_n < \lambda M \,.\, (k-1)$, $d_{n+1} < \lambda M\,(k-1)$ further

$$e^{M(k-1)} < p_n < e^{Mk} \text{ and } |d_{n+1} - d_n| < \frac{\varepsilon M k}{\lambda}$$

are valid, does not exceed

$$\frac{c_{10} \,.\, \varepsilon \,.\, e^{Mk}}{Mk}.$$

Proof of Lemma 3. The number of primes p_n for which $e^{M(k-1)} < p_n < e^{Mk}$, further $d_n = a$ and $d_{n+1} = b$ can be estimated by Lemma 1. If this number is denoted by $Z_{a,b}$, we have

$$(21) \qquad Z_{a,b} < \frac{c_9 e^{Mk}}{(Mk)^3} W(a)\, W(a+b).$$

If Z_k denotes the number of primes satisfying all conditions of Lemma 3, we obtain

$$(22) \qquad Z_k < \sum_{\substack{a < \lambda M(k-1),\ b < \lambda M(k-1) \\ |b-a| < \frac{\varepsilon k M}{\lambda}}} Z_{a,b}.$$

Using Lemma 2 we obtain

$$(23) \qquad Z_k < \frac{c_{10} \varepsilon e^{Mk}}{Mk}$$

for $k > k_0(M, \varepsilon, \lambda)$.

Let us proceed now to the proof of our theorem. According to (7) if we choose

$$M > \log \frac{4c_5}{c_4}$$

we have at least $\frac{c_4 e^{Mk}}{2Mk}$ primes in the interval $(e^{M(k-1)}, e^{Mk})$. As we have $\sum_{e^{M(k-1)} < p_n < e^{Mk}} d_n < e^{Mk}$ the number of primes p_n in $(e^{M(k-1)}, e^{Mk})$ for which $d_n \geqq \lambda M\,(k-1)$, does not exceed $\frac{e^{Mk}}{\lambda M\,(k-1)}$, and we have the same estimate for the number of primes p_n in the same interval for which $d_{n+1} \geqq \lambda M\,(k-1)$. Thus for at least

$$\left(\frac{c_4}{2} - \frac{4}{\lambda}\right)\frac{e^{Mk}}{Mk}$$

primes in $(e^{M(k-1)}, e^{Mk})$, we have both $d_n \leqq \lambda M\,(k-1)$ and $d_{n+1} < \lambda M\,(k-1)$. According to (23) the number of primes in $(e^{M(k-1)}, e^{Mk})$ for which $d_n < \lambda M\,(k-1)$, $d_{n+1} < \lambda M\,(k-1)$ and $|d_{n+1} - d_n| < \frac{\varepsilon k M}{\lambda}$ does not exceed $\frac{c_{10}\varepsilon\, e^{Mk}}{Mk}$. Let us choose now $\lambda = \frac{32}{c_4}$ and $\varepsilon = \frac{C_4}{8Mc_{10}}$. We obtain, that for at least $\frac{c_4 e^{Mk}}{4Mk}$ primes in $(e^{M(k-1)}, e^{Mk})$ we have $d_n < \lambda M\,(k-1)$, $d_{n+1} < \lambda M\,(k-1)$, and $|d_{n+1} - d_n| > \frac{\varepsilon k M}{\lambda}$.

Thus from (15) we obtain

$$(24)\quad D_N > \sum_{k_0 < k < \frac{\log N}{M}} \ \sum_{e^{M(k-1)} < p_n < e^{Mk}}{}' \frac{|d_{n+1} - d_n|}{p_n} > \frac{\varepsilon c_4}{4\lambda M} \sum_{k_0 < k < \frac{\log N}{M}} 1 > c_{11} \log N.$$

As it has been remarked above, this proves our theorem.

§ 3. In connection with the above estimations there arises the question, how many times it occurs for $p_n \leqq N$ that $d_{n+1} = d_n$, or more generally that $d_{n+1} = d_n + h$, where h is a fixed integer (positive, negative or zero). In this direction we prove

Theorem 2. The number of solutions of $p_n \leqq N$, $d_{n+1} = d_n + h$ does not exceed

$$(25)\qquad \frac{(\log N)^{3/2}}{c_{12}\, N}.$$

Proof of theorem 2. We have according to Lemma 1

$$(26)\quad H_N = \sum_{\substack{p_n \leqq N\ d_{n+1} = d_n + h \\ d_n < (\log N)^{3/2}}} 1 \leqq \frac{c_9 N}{\log^3 N} \sum_{m < (\log N)^{3/2}} W(m)\, W(m+h).$$

Applying the inequality of CAUCHY-SCHWARZ, we obtain

$$H_N < \frac{c_9 N}{\log^3 N} \sum_{m \leq (\log N)^{2/3} + h} W^2(m). \tag{27}$$

Now we need the following simple

Lemma 4.

$$\sum_{m=1}^{A} W^2(m) < c_{13} A. \tag{28}$$

Proof of Lemma 4. We have

$$\prod_{p=2}^{\infty} \frac{\left(1+\frac{1}{p}\right)^2}{\left(1+\frac{2}{p}\right)} = \prod_{p=2}^{\infty}\left(1+\frac{1}{p^2+2p}\right) = c_{14}$$

and thus

$$\sum_{m=1}^{A} W^2(m) \leqq c_{14} \sum_{m=1}^{A} \prod_{p/m}\left(1+\frac{2}{p}\right) = c_{14} \sum_{d=1}^{A} \frac{2^{V(d)}\,|\mu(d)|}{d} \Big(\sum_{rd \leq A} 1 \Big) < A c_{14} \sum_{d=1}^{\infty} \frac{2^{V(d)}}{d^2}$$

where $V(d)$ is the number of different prime factors of d. Denoting by $\tau(d)$ the number of divisors of d we have

$$2^{V(d)} \leqq \tau(d)$$

and thus

$$\sum_{d=1}^{\infty} \frac{2^{V(d)}}{d^2} \leqq \sum_{d=1}^{\infty} \frac{\tau(d)}{d^2} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^2 = \frac{\pi^4}{36}.$$

Thus we obtain

$$\sum_{m=1}^{A} W^2(m) < \frac{c_{14}\pi^4}{36} A$$

which proves Lemma 4. As $d_n \not< (\log N)^{2/3}$ can evidently occur not more than $N(\log N)^{-2/3}$ times, as it is seen from $\sum_{p_n \leq N} d_n \leqq N$, Theorem 2 is proved. 17

§ 4. We prove still some results concerning the sequence d_n. Recently W. SIERPINSKI proved the following theorem: [9])

For every positive integer K there exist an infinity of primes p with the property that all the numbers $p \pm k$, $k = 1, 2, \ldots K$ are composite i.e. that there are an infinity of primes which are "isolated" from both sides from the next prime by an interval of arbitrary large length. Evidently the theorem of SIERPINSKI is equivalent to $\lim\limits_{n\to\infty} \frac{1}{d_n} + \frac{1}{d_{n+1}} = 0$. The proof given by SIERPINSKI is based on the application of the theorem of DIRICHLET, that in every arithmetic progression $Dx + r$, $x = 1, 2, \dots$ $D, r = 1$ there are an infinity of primes. In what follows we shall show that a theorem which is stronger than that of SIERPRINSKI can be proved in an elementary way by using only BRUN's method. We shall prove

Theorem 3. For any integer N and any $r < c\sqrt{\log N}$ there is a prime $p_n \leqq N$ for which

$$d_{n+j} \geqq \frac{c_{15} \log N}{r^2}, \; j = 0, 1, 2, \dots (r-1).$$

Proof of theorem 3. We need the following

Lemma 5.

$$\sum_{m=1}^{A} \frac{W(m)}{m^\lambda} < \frac{3A^{1-\lambda}}{(1-\lambda)\, 2\lambda} \text{ for } 0 < \lambda < 1. \tag{29}$$

We have namely

$$\sum_{m=1}^{A} \frac{W(m)}{m^\lambda} = \sum_{d=1}^{A} \frac{1}{d^{1+\lambda}} \left(\sum_{r \leq \frac{A}{d}} \frac{1}{r^\lambda} \right) \leqq \frac{A^{1-\lambda}}{1-\lambda} \sum_{d=1}^{A} \frac{1}{d^{1+2\lambda}} \leqq$$

$$\leqq \frac{A^{1-\lambda}}{(1-\lambda)} \left(\frac{1}{2\lambda} + 1 \right) < \frac{3A^{1+\lambda}}{2\lambda\,(1-\lambda)}$$

which proves Lemma 5. Now we prove

Lemma 6.

$$\sum_{p_n \leq N} \frac{1}{d_n^\lambda} < \frac{c_{16} N}{\lambda\,(1-\lambda)\,(\log N)^{1+\lambda}}. \tag{30}$$

We apply the theorem of VIGGO BRUN mentioned in § 1 (see (3)). We obtain using also (7)

$$\sum_{p_n \leq N} \frac{1}{d_n^\lambda} \leqq \sum_{\substack{d_n \leq \log N \\ p_n \leq N}} \frac{1}{d_n^\lambda} + \tag{31}$$

$$+ \frac{1}{(\log N)^\lambda} \pi(N) \leqq \sum_{k \leq \log N} \frac{c N}{\log^2 N} \cdot \frac{W(k)}{k^\lambda} + \frac{c_5 N}{(\log N)^{1+\lambda}}.$$

Thus by Lemma 5 we obtain from (31)

$$\sum_{p_n \leq N} \frac{1}{d_n^\lambda} \leqq \frac{3c\, N (\log N)^{1-\lambda}}{\log^2 N\, \lambda\,(1-\lambda)} = \frac{3c\, N}{\lambda\,(1-\lambda)\,(\log N)^{1+\lambda}}. \tag{32}$$

which proves Lemma 6. Now Theorem 3 can be deduced easily. Let us put

$$\delta_n^{(r)} = \sum_{j=1}^{r-1} \frac{1}{\sqrt{d_{n+j}}}. \tag{33}$$

We have by Lemma 6 with $\lambda = \frac{1}{2}$.

$$\sum_{p_n \leq N-r+1} \delta_n^{(r)} \leqq r \sum_{p_n \leq N} \frac{1}{\sqrt{d_n}} < \frac{4c_{16} N \cdot r}{(\log N)^{3/2}}. \tag{34}$$

Thus if we put $\underset{p_n \leq N-r+1}{Min} \delta_n^{(r)} = \triangle_n^{(r)}$ we obtain from (34) using (7) that

$$\triangle_n^{(r)} < \frac{c_{17}\, r}{\sqrt{\log N}}. \tag{35}$$

Thus for an appropriate n we have

$$d_{n+j} > \frac{c_{18} \log N}{r^2} \text{ for } j = 0, 1, \ldots (r-1), \tag{36}$$

which proves Theorem 3. For $r = 2$ this theorem states, that there exists a constant C such that for every N there is a prime $p \leqq N$ for which all the numbers $p \pm k$, $k = 1, 2, \ldots [C \log N]$ are composite. This can be formulated also by saying, that the first prime having the property required by the theorem of SIERPINSKI, i.e. for which all numbers $p \pm k$, $k = 1, 2, \ldots K$ are composite, does not exceed e^{CK}.

We still add some remarks on Lemma 6. As it has been used before, for more than $\frac{cN}{\log N}$ primes $p_n \leqq N$ we have $d_n < K \log N$, and thus we obtain

$$\sum_{p_n \leq N} \frac{1}{d_n^{\lambda}} > \frac{c_{18} N}{(\log N)^{1+\lambda}}. \tag{37}$$

This means that the order of magnitude of the sum on the left of (30) given by Lemma 6 is exact. Now we can make the parameter λ in this sum depend on N, for example we may choose

$$\lambda = 1 - \frac{1}{\operatorname{loglog} N}.$$

We obtain from Lemma 6 that

$$\sum_{p_n \leq N} \frac{1}{d_n} < \frac{c_{19} N \operatorname{loglog} N}{\log^2 N}. \tag{38}$$

For the sum on the left of (38) we can give only the lower estimate

$$\sum_{p_n \leq N} \frac{1}{d_n} > \frac{c_{20} N}{(\log N)^2}. \tag{39}$$

As amatter of fact, if we could prove

$$\sum_{p_n \leqq N} \frac{1}{d_n} > \frac{cN\, \psi(N)}{(\log N)^2}$$

with $\psi(N) \to \infty$, it would follow that $\lim_{n\to\infty} \frac{d_n}{\log n} = 0$, but at present we are unable to prove anything of this type. To give some idea for the random distribution of the sequence d_n, we add a tabel giving the values of d_n for $p_n \leqq 4397$, $(n = 2, 3, \ldots 599)$.

Budapest, January 30, 1949.

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Table of the sequence d_n for $n = 2, 3, \dots 599$.

n	0,	1,	2,	3,	4,	5,	6,	7,	8,	9
0			1	2	2	4	2	4	2	4
10	6	2	6	4	2	4	6	6	2	6
20	4	2	6	4	6	8	4	2	4	2
30	4	14	4	6	2	10	2	6	6	4
40	6	6	2	10	2	4	2	12	12	4
50	2	4	6	2	10	6	6	6	2	6
60	4	2	10	14	4	2	4	14	6	10
70	2	4	6	8	6	6	4	6	8	4
80	8	10	2	10	2	6	4	6	8	4
90	2	4	12	8	4	8	4	6	12	2
100	18	6	10	6	6	2	6	10	6	6
10	2	6	6	4	2	12	10	2	4	6
20	6	2	12	4	6	8	10	8	10	8
30	6	6	4	8	6	4	8	4	14	10
40	12	2	10	2	4	2	10	14	4	2
50	4	14	4	2	4	20	4	8	10	8
60	4	6	6	14	4	6	6	8	6	12
70	4	6	2	10	2	6	10	2	10	2
80	6	18	4	2	4	6	6	8	6	6
90	22	2	10	8	10	6	6	8	12	4
200	6	6	2	6	12	10	18	2	4	6
10	2	6	4	2	4	12	2	6	34	6
20	6	8	18	10	14	4	2	4	6	8
30	4	2	6	12	10	2	4	2	4	6
40	12	12	8	12	6	4	6	8	4	8
50	4	14	4	6	2	4	6	12	6	10
60	20	6	4	2	24	4	2	10	12	2
70	10	8	6	6	6	18	6	4	2	12
80	10	12	8	16	14	6	4	2	4	2
90	10	12	6	6	18	2	16	2	22	6
300	8	6	4	2	4	8	6	10	2	10
10	14	10	6	12	2	6	2	10	12	2
20	16	2	6	4	2	10	8	18	24	4
30	6	8	16	2	4	8	16	2	4	8
40	6	6	4	12	2	12	6	2	6	4
50	6	14	6	4	2	6	4	6	12	6
60	6	14	4	6	12	8	6	4	26	18
70	10	8	4	6	2	6	22	12	2	16
80	8	4	12	14	10	2	4	8	6	6
90	4	2	4	6	8	4	2	6	10	2
400	10	8	4	14	10	12	2	6	4	2
10	16	14	4	6	8	6	4	18	8	10
20	6	6	8	10	12	14	4	6	6	2
30	18	2	10	8	4	14	4	8	12	6
40	12	4	6	20	10	2	16	26	4	2
50	12	6	4	12	6	8	4	8	22	2
60	4	2	12	18	2	6	4	6	4	6
70	2	12	4	12	2	10	2	16	2	16
80	6	20	16	8	4	2	4	2	22	8
90	12	6	10	2	4	6	2	6	10	2
500	12	10	2	10	14	6	4	6	8	6
10	6	16	12	2	4	14	6	4	8	10
20	8	6	6	22	6	2	10	14	4	6
30	18	2	10	14	4	2	10	14	4	8
40	18	4	6	2	4	6	2	12	4	20
50	22	12	2	4	6	6	2	6	22	2
60	6	16	6	12	2	6	12	16	2	4
70	6	14	4	2	18	24	10	6	2	10
80	12	10	2	10	6	2	10	2	10	6
90	8	30	10	2	10	8	6	10	18	6