A THEOREM ON THE DISTRIBUTION OF THE VALUES OF L-FUNCTIONS

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1. Let $(d/n)$ [where $d = o$, $1 \pmod{4}$, $d \neq u^2$, $u$ integral] be Kronecker’s symbol. Define for $s > 0$

$$L_d(s) = \sum_{n=1}^{\infty} \left( \frac{d}{n} \right)n^{-s}.$$

Denote by $g(d, x)$ the number of positive integers $d \leq x$ such that

$$d = o, 1 \pmod{4}; \quad d > o; \quad d \neq u^2; \quad L_d(s) < a.$$

We prove the following

**Theorem.** If $s > 3/4$ we have

$$\lim_{x \to \infty} \frac{g(a, x)}{x^{1/2}} = g(a)$$

exists; furthermore $g(0) = 0, g(\infty) = 1$ and $g(a)$, the distribution function, is a continuous and strictly increasing function of $a$.

It is implicit in our theorem that for almost all $d$ [i.e. with the exception of $o(x)$ integers $d \leq x$] $L_d(s) > 0$ provided that $s > 3/4$. This result seems to be new.

If the extended Riemann hypothesis holds, then of course $L_d(s) > 0$ for all $d$.

We can also prove our theorem when $d$ runs over negative integral values whose absolute values do not exceed $x$. (Similar questions on the distribution functions of number theoretic functions were considered in several papers of Wintner, e.g. *Amer. Jour. of Maths.* 63, (1941), 223-248; see also Jessen-Wintner, *Trans. Amer. Math. Soc.* 38 (1935), 48-88.)

2. Write for $s > 0$,

$$L_d(s, y) = \sum_{n \leq y} \left( \frac{d}{n} \right)n^{-s},$$

$$L_d'(s, y) = \sum_{n \leq y} \left( \frac{d}{n} \right)n^{-s}.$$
where, in the last two summations, \( n \) runs only over positive integers whose greatest prime factor does not exceed \( t \). Clearly

\[
L_d(s, x^{2/3}) - L_d(s, x^{2/3}) = \sum_{n<x^{2/3}, P(n)>t} \left( \frac{d}{n} \right) n^{-s},
\]

where \( P(n) \) denotes the greatest prime factor of \( n \). Hence

\[
\frac{1}{L_d(s, x^{2/3})} - \frac{1}{L_d(s, x^{2/3})} = \sum_{m, n<x^{2/3}, P(n)>t} \left( \frac{d}{m} \right) \left( \frac{d}{n} \right) (mn)^{-s}
\]

summing for all \( d \equiv 0, 1 \) (mod 4) which are in the range \( 2 \leq d \leq x \) and are not perfect squares:

\[
\sum_d |L_d(s, x^{2/3}) - L_d(s, x^{2/3})|^2 = \sum_d \left( \frac{d}{m} \right) \left( \frac{d}{n} \right) (mn)^{-s} = \Xi_1 + \Xi_2,
\]

where in \( \Xi_1 \) the product \( mn \) is not a perfect square, while in \( \Xi_2 \) the product \( mn \) is a square.

To estimate \( \Xi_1 \) we use

**Lemma 1.** If \( k \) is not a perfect square we have

\[
\sum_d \left( \frac{d}{k} \right) = O(k^{1/2} \log k) + O(x^{2/3}).
\]

The summation is for all \( d \) with

\[
2 \leq d \leq x; \ d \equiv 0, 1 \ (4); \ d \neq u^2.
\]

**Proof.** Polya (Gottinger Nachrichten, 1918) proved that

\[
\sum_{a} x(n) = O(k^{1/2} \log k)
\]

if \( x \) is a primitive character (mod \( k \), \( k \geq 1 \). From this several writers deduced that this result is true for any non-principal character (mod \( k \)). We easily deduce
which proves Lemma 1; the term $O(\sqrt{x})$ in the lemma is accounted for by the fact that the summation in our lemma excludes the $d$ which are perfect squares.

Thus using Lemma 1 we have (since $s > \frac{3}{4}$)

\[
|\Sigma_1| < c \sum_{m, n < x^{2/3}} \frac{\sqrt{(mn) \log (mn)} + \sqrt{x}}{(mn)^s} < c \left( x^{\frac{3}{2} - 2s} + x^{\frac{1}{2} - 2s} + \frac{1}{2} \right) = o(x). \tag{1}
\]

In $\Sigma_2$, $mn = w^2$, hence

\[
|\Sigma_2| \leq \sum_{w=1}^{x^{2/3}} \frac{xd(w^2)}{w^{2s}} < \frac{cx}{\sqrt{t}}, \tag{2}
\]

where $d(n)$ denotes the number of divisors of $n$. We used here the well-known fact that $d(n) = O(n^{\epsilon})$ for any $\epsilon > 0$ and also that $s > \frac{3}{4}$.

Thus we obtain from (1) and (2)

**Lemma 2.** Let $s > \frac{3}{4}$. Then there exists an absolute constant $c$ so that

\[
\sum_d |L_d(s, x^{\frac{2}{3}}) - L_d^{(1)}(s, x^{\frac{2}{3}})|^2 < \frac{cx}{\sqrt{t}}.
\]

3. We next estimate

\[
L_d(s) - L_d(s, x^{2/3}) = \sum_{n > x^{2/3}} \left( \frac{d}{n} \right) n^{-s} = O(\sqrt{d \log d}) = O(x^{\frac{1}{2} - 2s} + e) = o(1). \tag{3}
\]

Further we have

\[
L_d^{(1)}(s) - L_d^{(1)}(s, x^{\frac{2}{3}}) = \sum_{n > x^{2/3}} \left( \frac{d}{n} \right)^{n^{-s}} = O(\sum_{n > x^{2/3}} \left( \frac{d}{n} \right)^{n^{-s}}), \tag{4}
\]

since the number of integers $n \leq x$ for which $P(n) \leq t$ is less than

\[
\left( \frac{\log x}{\log 2} \right)^{\pi(t)} < \left( \frac{\log x}{\log 2} \right)^{\epsilon},
\]

where $\pi(t)$ is the number of primes less than $t$. 

\[\pi(t) = \sum_{n \leq t} \frac{1}{\log n} + O(1) = \log t + O(1).
\]
where \( \pi(y) \) denotes the number of primes \( \leq y \). We obtain by partial summation from (4) that
\[
| L_d^{(s)}(s) - L_d^{(2)}(s, \chi^2) | < c \left( \frac{\log x}{\log 2} \right) x^{-\frac{1}{2s}} = o(1),
\]
as \( x \) tends to \( \infty \), for any fixed \( t \).

Combining Lemma 2 with (3) and (5) we obtain

**Lemma 3.** For every positive \( s \) there exist an \( \epsilon \) and \( t_0 \) so that for the number of integers \( d \) with \( 1 < d \leq x \) \( [d = 0, 1 \ (4), \ d \neq u^2] \) satisfying
\[
\left| L_d^{(s)}(s) - L_d(s) \right| > \epsilon
\]
is \( \leq \delta x \) whenever \( t > t_0 \) and \( x \) is large enough \( [x > x_0(\delta)] \).

5. We define the primes
\( p_1 < p_2 < p_3 \ldots < p_k \)
and all less than \( t \) as follows:
\[ p_1 = 5; \ p_{i+1} \text{ is the least prime satisfying} \]
\[ p_{i+1} > 10 p_i \ (1 \leq i \leq k-1). \]
We define the signature of \( d \) with respect to a set of primes as the set of values of \( (d/p) \), where \( p \) runs through the given set of primes.

Denote by \( q_1, q_2, \ldots, q_m \) the primes \( \leq t \) which are distinct from the \( p \)'s. We have \( k+m = \pi(t) \).

Let \( d_1 \) and \( d_2 \) be two values of \( d \) which have different signatures with respect to the \( p \)'s but the same signature with respect to the \( q \)'s. Let \( p_i (j \leq k) \) be the first prime \( p \) for which the signatures of \( d_1 \) and \( d_2 \) disagree. Then we clearly have
\[
\text{Max} \left( \frac{L_{d_1}^{(s)}(s)}{L_{d_2}^{(s)}(s)} \right) > (1 + p_i^{-s}) \frac{k}{t=i+1} \left( \frac{1 - p_i^{-s}}{1 + p_i^{-s}} \right)
\]
\[
> (1 + p_i^{-s}) \left( 1 - 2 p_i^{-s} \right) > (1 + p_i^{-s}) \left\{ 1 - 2 \sum_{t=i+1}^{k} p_i^{-s} \right\}
\]
\[
> (1 + p_i^{-s}) \left\{ 1 - \frac{2}{p_j^s} \left( \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots \right) \right\}
\]
\[
= (1 + p_i^{-s}) \left( 1 - \frac{2}{9} p_i^{-s} \right)
\]
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$$= 1 + \frac{3}{2} p_j^{-s} - \frac{3}{2} p_j^{-2s} > 1 + \frac{3}{2} p_j^{-s} > 1 + \frac{1}{2} p_j^{-s}$$
$$\geq 1 + \frac{1}{2} p_j^{-s}.$$  \hspace{1cm} (6)

In the above we used that $2 p_j^{-s} < 1$ which follows from $s > \frac{3}{4}$, $p_1 = 5$. Also we used $p_{i+1} > 10 p_i$. We next prove

**Lemma 4.** If $a > 0$ and $0 < \varepsilon < \frac{a}{4 p_k + 1}$ the inequality

$$a - \varepsilon < L_{(2)}(s) < a + \varepsilon$$
cannot be satisfied by $d = d_1$ and $d = d_2$ if $d_1$ and $d_2$ are values of $d$ such that their signature with respect to the primes $p$ is different, while their signature with respect to the primes $q$ is the same.

From (6) it follows that Lemma 4 is true if $\varepsilon$ is so small that

$$\frac{a + \varepsilon}{a - \varepsilon} < 1 + \frac{1}{2 p_k^s},$$
$$\varepsilon < \frac{2^{-1} p_k^{-s}}{2 + 2^{-1} p_k^{-s}},$$
$$\varepsilon < \frac{a}{1 + 4 p_k^s}.$$

This proves the lemma.

Let $\gamma_s$ denote the number of $d \leq x \left[d > 0 ; d \equiv 0, 1 \left(4\right) ; d \neq u^2\right]$ such that all the $d$'s have a fixed signature with respect to the $q$'s. Clearly $s$ assumes

$$h = 3^n = 3^{\pi(x) - k}$$
values, and

$$\gamma_1 + \gamma_2 + \cdots + \gamma_h = \frac{x}{2} + O(\sqrt{x}),$$

where the constant implied in the $O$ is an absolute one.

Again the $d$'s (which have a fixed signature with respect to the $q$'s) fall into $3^e = g$ classes according to their signature with respect to the $p$'s. Thus

$$\gamma_s = z_{1s} + z_{2s} + \cdots + z_{gs} \left(1 \leq s \leq h\right).$$  \hspace{1cm} (7)

Clearly

$$gh = 3^{e(o)}.$$

Next we prove

**Lemma 5.** For $x > x_0 \left(k\right)$ we have
\[
\frac{z_{bs}}{y_s} < \frac{1}{2^b} \left[ 1 < b < 8 \right].
\]

**Proof.** The \(d's \leq x\) which have a particular signature with respect to the \(q's\) are (by assumption) \(y_s\) in number \((s = 1, 2, 3, \ldots, h)\). Let

\[q_{a1}, q_{a2}, \ldots, q_{aw}\]

be the primes \(q\) for which \((d/q) = 0\); let

\[q_{\beta 1}, q_{\beta 2}, \ldots, q_{\beta w'}\]

be the primes for which \((d/q) = +1\); finally let

\[q_{\gamma 1}, q_{\gamma 2}, \ldots, q_{\gamma w''}\]

be the primes \(q\) for which \((d/q) = -1\). We have

\[w + w' + w'' = \pi(t) - k = m.\]

It is evident that

\[
y_s = \frac{x}{2} \prod_{n=1}^{w} q_{a_n}^{-1} \prod_{n=1}^{w'} \left(\frac{q_{\beta_n} - 1}{2 q_{\beta_n}}\right) \prod_{n=1}^{w''} \left(\frac{q_{\gamma_n} - 1}{2 q_{\gamma_n}}\right) + O(\sqrt{x})
\]

\[
= \frac{Q x}{2} + O(\sqrt{x}), \quad (8)
\]

where the constant in the last \(O\) may also depend on \(t\). Consider next the value of \(z_{bs}\). This number is the number of \(d \leq x\) which have the above signature with respect to the \(q's\) and also have a fixed signature with respect to the \(p's\). Write

\[
\left(\frac{d}{p}\right) = 0 \text{ for } p = p_{a_n} (1 \leq n \leq v).
\]

\[
\left(\frac{d}{p}\right) = +1 \text{ for } p = p_{\beta_n} (1 \leq n \leq v').
\]

\[
\left(\frac{d}{p}\right) = -1 \text{ for } p = p_{\gamma_n} (1 \leq n \leq v'').
\]

Then

\[v + v' + v'' = k.\]

Further it is evident that

\[
z_{bs} = \frac{Q x}{2} \prod_{n=1}^{v} p_{a_n}^{-1} \prod_{n=1}^{v'} \left(\frac{p_{\beta_n} - 1}{2 p_{\beta_n}}\right) \prod_{n=1}^{v''} \left(\frac{p_{\gamma_n} - 1}{2 p_{\gamma_n}}\right)
\]

\[
= PQ \frac{x}{2} + O(\sqrt{x}). \quad (9)
\]
From (8) and (9), we have
\[ z_{b_2}/y_s = P + O(x^{-\delta}). \]  
(10)
The lemma thus follows since \( P < 2^{-k} \) \([k > 1]\).

Consider now the \( d \)'s which have the same signature as the numbers of \( y_s \) \((1 < s < h)\). By Lemma 4 at most 
\[ \max (z_{b_i}) \leq b < 3^k \] of them satisfy the inequality 
\[ a - \varepsilon < L_{d_d}^0 (s) < a + \varepsilon. \]  
(11)
Hence by Lemma 5 the total number of \( d \)'s not exceeding \( x \) which satisfy (11) is at most 
\[ z_{a_1} + z_{a_2} + z_{a_3} + \ldots < 2^{-k} (y_1 + y_2 + y_3 + \ldots) = 2^{-k} \left( \frac{x}{2} + O(\sqrt{x}) \right). \]
Thus, we have (choose \( 2^{-k} \approx \delta \)) 

**Lemma 6.** Given any positive \( \delta \), there exist \( t_0, \varepsilon, x_0 \) such that the number of positive \( d \leq x \) with \( d = o, d = (4), d \neq u^2, and \)
\[ a - \varepsilon < L_{d_d}^0 (s) < a + \varepsilon \]  
(12)
is less than \( \delta x \) for all \( t > t_0, x > x_0 \).

The case \( a = o \) needs special discussion. Here (12) has to be replaced by 
\[ 0 < L_{d_d}^o (s) < \varepsilon \]  
(13) 
whence
\[ S_d = \prod_{d \leq x} \{ 1 - (d/p) p^{-s} \} > \varepsilon^{-1}. \]
Now the sum
\[ S_d = x/2 + O(\sqrt{x}), \]
where \( d \) runs over integers which are \( o, (4), not \) perfect squares, and \( \leq x \).

It easily follows that Lemma 6 is true with \( a = o \) when we replace (12) by (13).

6. **Proof of the theorem.** Denote by \( g_t (a, x) \) the number of integers \( d \leq x \) \([d = o, (4), d \neq u^2, d > o]\) for which
\[ L_{d_d}^0 (s) \leq a. \]
It is easy to see that
\[
\lim_{x \to \infty} \frac{g_t(a, x)}{x/2} = g(a, t)
\]
eexists. This follows from the following simple observation. The expression
\[
\prod_{\substack{t < \sigma \leq \tau \leq t \leq \tau + 1 \leq \sigma \leq \tau}} \left\{ 1 - \left( \frac{d}{p} \right) \right\}^{-1}
\]
is periodic in \( d \pmod{p_1 p_2 \ldots p_w} \), where \( p_1, \ldots, p_w \) are all the primes \( \leq t \). As \( d \) goes from 1 to \( p_1 p_2 \ldots p_w \) suppose that there are \( N_t \) values of \( d \) for which
\[
0 < L_{d}^{(t)}(s) < a.
\]
Then
\[
\lim_{x \to \infty} \frac{g_t(a, x)}{x/2} = \frac{2N_t}{p_1 p_2 \ldots p_w}.
\]
We next prove
\[
\lim_{x \to \infty} g(a, t) = g(a),
\]
where \( g(a) \) was defined in \( \S \, 1 \).

To do this it will suffice to show that given an arbitrary positive \( \eta \) we can find \( t_0, \sigma_0 \) such that
\[
|g_t(a, x) - g(a, x)| < \eta x
\]
for \( t > t_0, x > \sigma_0 (\eta) \).

We split the integers \( d \leq x \) \([d = 0, 1 (d), d \neq u^2]\) which satisfy
\[
L_{d}^{(t)}(s) < a, L_d(s) > a
\]
or
\[
L_{d}^{(t)}(s) > a, L_d(s) < a
\]
in two classes:

I \[ |L_{d}^{(t)}(s) - L_d(s)| > \epsilon. \]

By Lemma 3 the number of these integers is \( < \delta x \).

II \[ a - \epsilon < L_{d}^{(t)}(s) < a + \epsilon. \]

By Lemma 6 the number of these integers is \( < \delta x \).

This completes the proof of our theorem. The fact that \( g(a) \) is a continuous and strictly increasing function of \( a \) follows easily by the arguments of Lemmas 3 and 6.

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