ON A TAUBERIAN THEOREM FOR EULER SUMMABILITY

by

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Let $\Sigma a_n$ be an infinite series. Put

$$a'_n = \frac{1}{2^{n+1}} \left( \sum_{k=0}^{n} \binom{n}{k} a_k \right).$$

(1)

$\Sigma a'_n$ is said to be the Euler sum of $\Sigma a_n$. It is easy to see that $\Sigma a'_n$ converges if $\Sigma a_n$ converges, but the converse is not true. Euler summability was first studied by Knopp.\(^3\)

W. Meyer-König proved\(^3\) that if $\Sigma a_n$ is Euler summable and $a_n = 0$ except if $n = n_i$, $n_{i+1}/n_i > c > 1$, then $\Sigma a_n$ is convergent. He also conjectured that the conclusion of the theorem would follow from the following weaker condition: $a_n = 0$ except if $n = n_i$, where $n_{i+1} - n_i > c n_i^{1/2}$, $c > 0$ any constant. In fact he proved\(^4\) this conjecture under the further assumption that $|a_n| < n^\alpha$ where $\alpha$ is any constant. It is easy to see that this conjecture is true if $f(n)$ tends to infinity arbitrarily slowly there exists a series $\Sigma w_n$ which is Euler summable but not convergent and for which $a_n = 0$ except if $n = n_i$, $n_{i+1} - n_i > n_i^{1/2}/f(n_i)$.

\(^1\) (1) gives a series to series transformation method. The corresponding sequence to sequence method would be

$$s'_n = \frac{1}{2^{n+1}} \left( \sum_{k=0}^{n} \binom{n}{k} s_k \right).$$

The two methods are equivalent, but for our present purpose the series to series transformation seems to be more suitable.

In the present note we are going to prove the following

**Theorem.** There exists a constant $A > 0$ so that if $\Sigma a_n$ is a series which is Euler summable, and for which $a_n = 0$ except if $n = n_i$

$$n_{i+1} - n_i > A n_i^{1/2},$$

then $\Sigma a_n$ is convergent.

At present I am unable to decide whether $A$ can be any constant greater than 0, in other words I am unable to prove Meyer-König's conjecture.

Let $\left( \frac{n}{m} \right) a_m/2^{n+1}$ be the summand of greatest absolute value in (1). If there are several such terms we consider the one with the greatest index $m$. Put $m = f(n)$.\footnote{Math. Zeitschrift 45 (1939), p. 479—494.}

**Lemma 1.** $f(n)$ is a non-decreasing function of $n$.

To prove lemma 1 it will clearly suffice to show that if

$$\left( \frac{n}{m} \right) a_m > \left( \frac{n-1}{l} \right) a_l$$

for $m > l$, then

$$\left( \frac{n}{m} \right) a_m > \left( \frac{n-1}{l} \right) a_l$$

This is true since

$$\left( \frac{n+1}{m} \right) a_m > \left( \frac{n}{m} \right) a_m > \left( \frac{n-1}{l} \right) a_l$$

i.e.

$$\frac{n+1}{n-m+1} > \frac{n+1}{n-l+1} \text{ for } m > l.$$ 

**Lemma 2.** Assume that $f(n) > n/2$. Then $F(n+1) > F(n)$.

Put $f(n) = m > n/2$. We obtain from $F(n+1) > \frac{1}{2^{n+2}} \left( \left( \frac{n+1}{m} \right) a_m \right) A_{n+1}$

$$F(n+1)/F(n) > \left( \frac{n+1}{m} \right) / 2 \left( \frac{n}{m} \right) = \frac{n+1}{2(n-m+1)} > 1,$$

which proves the lemma.

**Lemma 3.** Let $\alpha$ be arbitrary. Assume that $|a_n| < n^\alpha$ for all $n$, and $a_n = 0$ except if $n = n_i, n_{i+1} - n_i > c n_i^{1/2}$, where $c > 0$ is an arbitrary positive constant. Then if $\Sigma a_n$ is Euler summable it is convergent.

This is a theorem of Meyer-König.\footnote{Math. Zeitschrift 45 (1939), p. 479—494.}

Because of lemma 3 we can now assume that, for infinitely many $n$, $|a_n| > n$. We shall show that if an infinite series satisfies (2) and
\[ |a_n| > n \text{ for infinitely many } n \text{ then it cannot be Euler summable. This together with lemma 3 will complete the proof of our theorem. First we prove} \]

**Lemma 4.** Let \( c_1 > 0 \) be suitable constant. Then there exist infinitely many integers \( n \) satisfying

\[ n/2 \leq f(n) - f(n+1) = \cdots = f(n+t), \quad t \geq \frac{A}{3} n^{1/3} \]  

and

\[ F(n) > c_1 n^{1/3}, \quad F(n+1) > c_1 n^{1/3}, \ldots, F(n+t) > c_1 n^{1/3}. \]  

First of all it is easy to see that there exist infinitely many integers \( n_i \) satisfying

\[ \left| a_{n_i} \right| > n_i, \quad \left| a_k \right| < \left| a_{n_i} \right| \quad \text{for } 1 \leq k < n_i. \]  

To prove (5) it suffices to choose \( |a_n| > n \) and define \( a_{n_i} \) as the \( a_k \) of largest absolute value for \( 1 \leq k \leq n \).

Put

\[ a_{2n_i} = \frac{1}{2^{2n_i+1}} \sum_{k=0}^{2n_i} \binom{2n_i}{k} a_k. \]

By the second inequality of (5) we have \( f(2n_i) \geq n_i \), and by the first inequality of (5) for sufficiently large \( n_i \)

\[ F(2n_i) \geq \left| \binom{2n_i}{n_i} a_{n_i}/2^{2n_i+1} \right| > n_i \left| \binom{2n_i}{n_i}/2^{2n_i+1} \right| > c_2 n_i^{1/3}. \]  

Assume first that for infinitely many \( n \) satisfying (5) we have \( f(2n_i) = n_i \). From lemma 1 we have for \( x \geq 0 \)

\[ f(2n_i - x) \leq f(2n_i) = n_i. \]

Further, a simple argument shows that for \( t \leq n_i - n_i - 1 \) and \( j \geq 1 \)

\[ \binom{2n_i-t}{n_i} \]  

Therefore by the second inequality of (5)

\[ \left| \binom{2n_i-t}{n_i} a_{n_i} \right| > \left| \binom{2n_i-t}{n_{i-j}} a_{n_{i-j}} \right|. \]  

(7) and (9) imply that for \( 0 \leq j \leq n_i - n_i - 1 \)

\[ f(2n_i - t) = f(2n_i) = n_i. \]
A simple computation gives that for $t < A n_i^{1/2}$

$$\binom{2n_i-t}{n_i} > c_3 \binom{2n_i}{n_i}/2^t. \quad (11)$$

Therefore from (2)\(^5\) (6) and (11) we have for $t < A n_i^{1/2}$

$$F(2n_i-t) \geq \frac{1}{2^{2n_i-t+1}} \binom{2n_i-t}{n_i} |a_{n_i}| > c_3/2^{2n_i+1} \binom{2n_i}{n_i} |a_{n_i}| > c_2 c_3 n_i^{1/2} > 1 \quad (12)$$

(10) and (12) prove our lemma.

Assume next that for all sufficiently large $n_i$ satisfying (5) we have $f(2n_i) > n_i$. Put $n_1 = n_0$, $f(2n_0) = n_1$, $f(2n_1) = n_2$ . . . . There thus exists an infinite sequence $n_0, n_1, . . . . \ $ such that

$$n_i < n_1 < . . . , \ 2n_r \geq n_{r+1}, \ f(2n_r) = n_{r+1},$$

To simplify the notation we shall write $n_r$ instead of $n_i$, whenever there is no danger of confusion. First of all we show that all the $n_r$ satisfy (5). We use induction. By assumption $n_0$ satisfies (5). Assume that $n_r$ satisfied (5). A simple computation gives for sufficiently large $A$

$$\binom{2n_r}{n_r+1} < \binom{2n_r}{n_r + A [n_r]^{1/2}} < \frac{1}{2} \binom{2n_r}{n_r}. \quad (13)$$

Thus

$$\left| \binom{2n_r}{n_r+1} a_{n_r+1} \right| > \left| \binom{2n_r}{n_r} a_{n_r} \right|$$

implies

$$|a_{n_r+1}| > 2 |a_{n_r}| > 2 n_r \geq n_{r+1}$$

which is the first inequality of (5). Further since the binomial coefficients $\binom{2n_r}{n_r+1}$ decrease as $l$ increases, it follows from $f(2n_r) = n_{r+1}$ that

$$|a_{n_{r+1}}| > |a_n| \quad \text{for} \quad n_r < n < n_{r+1}.$$

But then since $n_r$ satisfied the first inequality of (5) it clearly follows that $n_{r+1}$ also satisfied it, which completes our proof.

Next we prove that for all $n \geq 2n_0$

$$F(n) > c_4 n^{1/2}. \quad (14)$$

\(^5\) This is the only place where our assumption that $A$ is sufficiently large is essential.
On a Tauberian theorem for Euler summability

From (13) and lemma 1 it follows that for \( n > 2 n_0 \), \( f(n) > n/2 \). Hence we have from lemma 2 that for \( n > 2 n_0 \) \( F(n) \) is an increasing function of \( n \). Let

\[ 2n_r \leq n < 2n_{r+1} \leq 4n_r. \]

Since \( n_r \) satisfies (5) we have

\[ F(n) \geq F(2n_r) \geq \left( \frac{2n_r}{n_r} \right) a_{n_r} > c_6 n_r^{1/2} > c_7 n_r^{1/2} \text{ q.e.d.} \]

Consider now the interval \( 2n_i \leq n \leq 4n_i \). Clearly \( n_i < f(n) < 4n_i \). Also \( f(n) \) must be one of the \( n_j \)'s. But by (2) the difference of two consecutive \( n_i \)'s is greater than \( A n_i^{1/2} \), (\( n_j > n_i \)). Thus the number of \( n_j \)'s in the interval \((n_i, 4n_i)\) is less than \( 3n_i^{1/2}/A \). Hence there must be at least

\[ 2n_i \left( \frac{3n_i}{A} \right) \geq \frac{2A}{3} n_i^{1/2} \]

integers in the interval \((n_i, 4n_i)\) with the same \( f(n) \) and by Lemma 1 they must be consecutive integers say \( n, n+1, \ldots, n+t \ t > A/3 n_i^{1/2} \). Thus (14) completes the proof of Lemma 4.

Now we can prove our theorem. Let \( n \) satisfy lemma 4 and choose

\[ t = \left\lceil \frac{A}{3} n_i^{1/2} \right\rceil + 1. \]

Put \( \left\lceil \frac{2n+t}{2} \right\rceil = M \). We have \( a'_M = \frac{1}{2^{M+1}} \sum_{k=0}^{M} \binom{M}{k} a_k \).

We shall show that \( |a'_M| > c_8 M^{1/2} \) where \( c_8 \) is an absolute constant independent of \( n \). This will of course show that \( \Sigma a'_n \) can not converge, hence \( \Sigma a_n \) was not Euler summable and the proof of our theorem will be complete.

Put \( f(M) = n_j \). We have by (4)

\[ F(1) = \frac{1}{2^{M+1}} \left| \binom{M}{n_j} a_{n_j} \right| > c_1 n_i^{1/2} > c_1/2 M^{1/2}. \]

We have

\[ |a'_M| \geq \frac{1}{2^{M+1}} \left[ \left( \binom{M}{n_j} a_{n_j} \right) - \sum_{n_r > n_j} \binom{M}{n_r} a_{n_r} - \sum_{n_r < n_j(n_r)} \binom{M}{n_r} a_{n_r} \right] = \]

\[ \frac{1}{2^{M+1}} \left[ \left( \binom{M}{n_j} a_{n_j} \right) - \Sigma_1 - \Sigma_2 \right]. \]

(16)

For an estimate of \( \Sigma_2 \) put \( r - j = k \), then \( n_r - n_j > A n_i^{1/2} \). Put

\[ n + t = M + x, \quad A n_i^{1/2} \leq x \leq \frac{A}{6} n_i^{1/2} + 1. \]
We have by
\[ f(n) = f(n+t) = n_j \ll x > \frac{A}{12} M^{1/2} \]
\[ \left| \binom{M+x}{n} a_{n_r} \right| \leq \left| \binom{M+x}{n} a_{n_j} \right| \]
Hence
\[ \left| \binom{M}{n} a_{n_r} \right| \leq \left| \binom{M}{n_j} a_{n_j} \right| \left| \frac{M-n_r+1}{M-n_j+1} \cdot \frac{M-n_r+2}{M-n_j+2} \cdots \frac{M-n_r+x}{M-n_j+x} \right| < \]
\[ \left( \binom{M}{n_j} a_{n_j} \right) \left( 1 - \frac{kA(n_j)^{1/2}}{M} \right)^x \left( \binom{M}{n_j} a_{n_j} \right) \left( 1 - \frac{kA(M^2/2)^{1/2}}{M} \right)^{A/12} M^{1/2} \]
\[ \leq \left( \binom{M}{n_j} a_{n_j} \right) \left( 1 - \frac{kA}{2M^{1/2}} \right)^{A/12} M^{1/2} \leq \left( \binom{M}{n_j} a_{n_j} \right) e^{-kA^2/24} \]
since from \( f(M) = n_j, n_j \gg M/2 \) (lemma 4). Thus for sufficiently large \( A \)
\[ \sum_1 < \left| \binom{M}{n_j} a_{n_j} \right| \sum_{k=1}^{\infty} e^{-kA^2/24} < \frac{1}{4} \left| \binom{M}{n_j} a_{n_j} \right| . \quad (17) \]
In the same way we can show
\[ \sum_2 < \frac{1}{4} \left| \binom{M}{n_j} a_{n_j} \right| . \quad (18) \]
Thus by (15), (16), (17) and (18)
\[ \left| a_M^r \right| > \frac{1}{2} \left| \binom{M}{n_j} a_{n_j} \right| / 2^{M+1} = \frac{1}{2} F(M) > \frac{c_1}{2} M^{1/2} \]
which completes the proof of the theorem.

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