Denote by $\Sigma_{n,s}$ the sum of the products of the first $n$ natural numbers taken $s$ at a time, i.e. the $s$-th elementary symmetric function of $1, 2, \ldots, n$. Hammersley† conjectured that the value of $s$ which maximises $\Sigma_{n,s}$ for a given $n$ is unique. In the present note I shall prove this conjecture and discuss some related problems.

We shall denote by $f(n)$ the largest value of $s$ for which $\Sigma_{n,s}$ assumes its maximum value. As Hammersley† remarks, it follows immediately from a theorem of Newton that

$$\Sigma_{n,1} < \Sigma_{n,2} < \cdots < \Sigma_{n,f(n)-1} \leq \Sigma_{n,f(n)} > \Sigma_{n,f(n)+1} > \cdots > \Sigma_{n,n} = n!.$$  \hspace{1cm} (1)

Thus it follows from (1) that the uniqueness of the maximising $s$ will follow if we can prove that

$$\Sigma_{n,f(n)-1} < \Sigma_{n,f(n)}.$$ \hspace{1cm} (2)

Hammersley proves (2) for $1 \leq n \leq 188$. He also proves that

$$f(n) = n - \left[ \log (n+1) + \gamma - 1 + \frac{\xi(2)-\xi(3)}{\log (n+1) + \gamma - \frac{3}{2}} + \frac{h}{(\log (n+1) + \gamma - \frac{3}{2})^2} \right],$$ \hspace{1cm} (3)

\* Received 27 February, 1952; read, 20 March, 1952.
where \([x]\) denotes the integral part of \(x\), \(\gamma\) denotes Euler's constant, \(\zeta(k)\) is the Riemann \(\zeta\)-function and \(-1.1 < h < 1.5\). Thus for \(n > 188 > e^5\) we obtain by a simple computation
\[
[\log n - \frac{1}{2}] \leq n - f(n) \leq [\log n].
\] (4)

First we prove

**Theorem 1.** For sufficiently large \(n\) all the integers \(\Sigma_{n,s}, 1 \leq s \leq n, \) are different.

We evidently have*
\[
\Sigma_{n,n-k} < \frac{n!}{k!} \left( \sum_{l=1}^{n} \frac{1}{l} \right)^k < \frac{n!}{k!} (1+\log n)^k < n! \left( \frac{e}{k} (1+\log n) \right)^k \leq n! = \Sigma_{n,n} \quad (5)
\]
for \(k \geq e(\log n+1)\). Thus from (1) and (5) it follows that to prove Theorem 1 we have only to consider the values
\[
0 \leq k < e(\log n+1).
\] (6)

The Prime Number Theorem in its slightly sharper form states that for every \(l\)
\[
\pi(x) = \int_{2}^{x} \frac{dy}{\log y} + O \left( \frac{x}{(\log x)^{2}} \right).
\] (7)

From (7) we have that for sufficiently large \(x\) there is a prime between \(x\) and \(x+x/(\log x)^2\). Thus we obtain that for \(n > n_0\) and \(k < e(\log n+1)\) there always is a prime \(p_k\) satisfying
\[
\frac{n}{k+1} < p_k \leq \frac{n}{k}.
\]

We have
\[
\Sigma_{n,n-k} \equiv 0 \quad (\text{mod} \ p_k).
\] (8)

For \(\Sigma_{n,n-k}\) is the sum of \(\binom{n}{k}\) products each having \(n-k\) factors. Clearly only one of these products is not a multiple of \(p_k\) (viz., the one in which none of the \(k\) multiples not exceeding \(n\) of \(p_k\) occur); thus (8) is proved.

For \(r < k\) all the \(\binom{n}{r}\) summands of \(\Sigma_{n,n-r}\) are multiples of \(p_k\). Thus
\[
\Sigma_{n,n-r} \equiv 0 \quad (\text{mod} \ p_k).
\] (9)

(8) and (9) complete the proof of Theorem 1.

We now give an elementary proof of Theorem 1 which will be needed in the proof of Hammersley's conjecture. Let
\[
r < k < e(\log n+1).
\] (10)

* The proof is similar to the one in a joint paper with Niven, Bull. Amer. Math. Soc., 52 (1946), 248-251. We prove there that for \(n > n_0\), \(\Sigma_{n,s} \equiv 0 \quad (\text{mod} \ n!)\).
We shall prove that for \( n > 10^8 \)
\[
\Sigma_{n, n-r} \neq \Sigma_{n, n-k}.
\] (11)

Let \( q \) be a prime satisfying \( n/2k < q \leq n/k \). Assume that
\[
\frac{n}{l+1} < q \leq \frac{n}{l}, \quad k \leq l \leq 2k-1.
\]

Clearly
\[
\Sigma_{n, n-r} \equiv 0 \pmod{q^{l-r}}. \] (12)

Now we compute the residue of \( \Sigma_{n, n-k} \pmod{q^{l-k+1}} \). Clearly
\( \Sigma_{n, n-k} \equiv 0 \pmod{q^{l-k}} \). The only summands of \( \Sigma_{n, n-k} \) which are not multiples of \( q^{l-k+1} \) are those which contain \( \Pi't \) where the product is extended over the integers \( 1 \leq t \leq n \), \( t \equiv 0 \pmod{q} \). \( \Pi't \) contains \( n-l \) factors, and the remaining \( l-k \) factors of the summands in question of \( \Sigma_{n, n-k} \) must be among the integers \( q, 2q, \ldots, lq \). Thus clearly
\[
\Sigma_{n, n-k} \equiv \Sigma_{l, l-k} \cdot \Pi't \cdot q^{l-k} \pmod{q^{l-k+1}}. \] (13)

Therefore if (11) does not hold we must have
\[
\Sigma_{l, l-k} \equiv 0 \pmod{q} \quad \text{(i.e. \( \Sigma_{n, n-k} \equiv \Sigma_{n, n-r} \equiv 0 \pmod{q^{l-k+1}} \))}.
\]

Thus if (11) is false
\[
\Pi_{n/2k < q \leq n/k} q \overset{2k-1}{\mid} \Pi_{l=k} \Sigma_{l, l-k}.
\] (14)

Now evidently (we can of course assume that \( k \geq 2 \) for if \( k = 1 \) then (11) clearly holds)
\[
\Pi_{l=k} \Sigma_{l, l-k} < \Pi_{l=k} (\frac{1}{k}) l-k < \Pi_{l=k} (2k)^l < (2k)^{2k} < k^{3k^2} < (3 \log n)^{27(\log n)^2}, \]
(15)
since for \( n > 10^8 > e^{10} \), \( k < e(1+\log n) < 3 \log n \). Define
\[
\vartheta(x) = \Sigma_{p \leq x} \log p.
\]

By the well-known results of Tchebycheff* we have
\[
\vartheta(2x) - \vartheta(x) > 0.7 \cdot x - 3.4 \cdot x^4 - 4.5(\log x)^2 - 24 \log x - 32.
\]

Thus for \( n > 10^4 \) we have by a simple computation
\[
\vartheta(2x) - \vartheta(x) > \frac{1}{2}x. \] (16)

For \( n > 10^8 \), we have \( n/2k > n/(6 \log n) > 10^4 \). Thus from (16) we have
\[
\Pi_{n/2k < q \leq n/k} q \overset{e^{n/4k}}{>} \overset{e^{n/(12 \log n)}}{>.}
\] (17)

---

From (14), (15) and (17) we have

$$(3 \log n)^{27(\log n)^3} \geq \frac{n}{(12 \log n)}.$$

Thus on taking logarithms and using $\log (3 \log n) < \log n$ for $n > 10^8$,

$$27 (\log n)^3 > \frac{n}{(12 \log n)} \quad \text{or} \quad 324 (\log n)^4 > n,$$

which is false for $n > 10^8$. Thus the proof of Theorem 1 is complete.

**Theorem 2** (Hammersley’s conjecture). The value of $s$ which maximises $\Sigma_n, s$ is unique; in other words

$$\Sigma_{n, f(n)} \neq \Sigma_{n, f(n)}.$$  \hspace{1cm} (18)

It follows from the second proof of Theorem 1 that Theorem 2 certainly holds if for $n > 10^8$. Thus since Hammersley proved Theorem 2 for $n \leq 188$ it suffices to consider the interval $188 < n < 10^8$.

Put $n - f(n) = t$. We have, from (4),

$$\log n - 2 < t \leq \log n.$$  \hspace{1cm} (19)

As was shown in the first proof of Theorem 1, (18) certainly holds if there is a prime satisfying

$$\frac{n}{t+2} < p \leq \frac{n}{t+1}.$$  \hspace{1cm} (20)

It follows from (19) that if $1500 < n < 10^8$

$$150 < \frac{n}{t+2} < 10^7.$$  \hspace{1cm} (21)

The tables of primes* show that for $150 < x < 10^7$ there always is a prime $q$ satisfying $x < q < x + x^4$. For $n > 1500$ we have

$$\frac{n}{t+2} + \left(\frac{n}{t+2}\right)^4 < \frac{n}{t+1},$$

since

$$\frac{n}{(t+1)(t+2)} > \left(\frac{n}{t+2}\right)^4,$$

or, by using (19),

$$n > (1 + \log n)^2 (2 + \log n),$$

which holds for $n > 1500$. Thus for $1500 < n < 10^7$ there always is a prime in the interval (20) and thus Theorem 2 is proved for $n > 1500$.

To complete our proof we only have to dispose of the $n$ satisfying $188 < n \leq 1500$. Hammersley† showed that for $n \leq 1500$ the only doubtful values of $n$ are: $189 < n < 216$, $539 < n < 580$. He also showed that if $189 < n < 216$ and (18) does not hold, then $t = 5$. But then $p = 31$ is in the interval (20), which shows that (18) holds in this case. If $539 < n < 580$ and (18) does not hold, he shows that $t = 6$. But then

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† See footnote †, p. 232.
either \( p = 73 \) or \( p = 79 \) lies in the interval \((20)\). Thus (18) holds here too, and the proof of Theorem 2 is complete.

By slightly longer computations we could prove that for \( n \geq 5000 \) Theorem 1 holds. Theorem 1 is certainly not true for all values of \( n \) since \( \Sigma_{2,1} = \Sigma_{3,3} \). Hammersley proved that for \( n \leq 12 \) this is the only case for which Theorem 1 fails, and it is possible that Theorem 1 holds for all \( n > 3 \). The condition \( n \geq 5000 \) could be considerably relaxed, but to prove Theorem 1 for \( n > 3 \) would require much longer computations.

Let \( u_1 < u_2 < \ldots \) be an infinite sequence of integers. Denote again by \( \Sigma_{n,s} \) the sum of the products of the first \( n \) of them taken \( s \) at a time. It seems possible that for \( n > n_0 \) (\( n_0 \) depends on the sequence) the maximising \( s \) is unique and even that for \( n > n_1 \) all the \( n \) numbers \( \Sigma_{n,s} \), \( 1 \leq s \leq n \) are distinct. If the \( u \)'s are the integers \( \equiv a \pmod{d} \) it is not hard to prove this theorem.

Stone and I proved by elementary methods the following

\textbf{Theorem.} Let \( u_1 < u_2 < \ldots \) be an infinite sequence of positive real numbers such that

\[ \sum \frac{1}{u_i} = \infty \quad \text{and} \quad \sum \frac{1}{u_i^3} < \infty. \]

Denote by \( \Sigma_{n,s} \) the sum of the product of the first \( n \) of them taken \( s \) at a time and denote by \( f(n) \) the largest value of \( s \) for which \( \Sigma_{n,s} \) assumes its maximum value. Then

\[ f(n) = n - \left[ n \sum_{i=1}^{n} \frac{1}{u_i} - \sum_{i=1}^{n} \frac{1}{u_i^3} \left( 1 + \frac{1}{u_i} \right)^{-1} + o(1) \right]. \]

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