Integral Functions with Gap Power Series

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1. Let

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]  \hspace{1cm} (1)

be an integral function, \( \lambda_n \) being a strictly increasing sequence of non-negative integers. We shall use the notations

\[ M(r) = \max |f(z)|, \quad m(r) = \min |f(z)|, \]
\[ \mu(r) = \max \{a_n | r^n\}, \]

describing \( M(r) \) as the maximum modulus, \( m(r) \) as the minimum modulus and \( \mu(r) \) as the maximum term of \( f(z) \).

The present paper is a development of a remark by Pólya (Math. Zeit., 29 (1929), 549-640, last sentence of the paper) that if

\[ \lim \log (\lambda_{n+1} - \lambda_n) > \frac{1}{2} \]  \hspace{1cm} (2)

then

\[ \lim_{r \rightarrow \infty} \frac{m(r)}{M(r)} = \lim_{r \leftarrow \infty} \frac{\mu(r)}{M(r)} = 1. \]  \hspace{1cm} (3)

Our first result is

**Theorem 1.**

If

\[ \sum_{n=0}^{\infty} \frac{1}{\lambda_{n+1} - \lambda_n} < \infty, \]  \hspace{1cm} (4)

then (3) holds.

Theorem 1 is clearly a sharpened form of Pólya's result, for from (2) it evidently follows that for sufficiently large \( n \)

\[ \lambda_{n+1} - \lambda_n > \lambda_n^{1+\epsilon} > n^{1+\delta} \]

for some positive \( \epsilon \) and \( \delta \).

Theorem 1 is best possible, as is shown by our next result.

**Theorem 2.**

If

\[ \sum_{n=0}^{\infty} \frac{1}{\lambda_{n+1} - \lambda_n} = \infty, \]  \hspace{1cm} (4)
then there exists an integral function of the form (1) such that

$$\lim_{r \to \infty} \frac{\mu(r)}{M(r)} = \frac{1}{2}, \quad \lim_{r \to \infty} \frac{m(r)}{M(r)} \leq \frac{1}{2}. \quad (6)$$

We generalise these theorems in two ways. First, relaxing the gap hypothesis we have

**Theorem 3.**

*If for a positive integer $h$*

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+h} - \lambda_n} < \infty \quad (7)$$

*then*

$$\lim_{r \to \infty} \frac{\mu(r)}{M(r)} \geq \frac{1}{2h-1}; \quad (8)$$

*but if*

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+h} - \lambda_n} = \infty \quad (9)$$

*for every $h$, then there exists an integral function of the form (1) such that*

$$\lim_{r \to \infty} \frac{\mu(r)}{M(r)} = \lim_{r \to \infty} \frac{m(r)}{M(r)} = 0. \quad (10)$$

The conjecture that under condition (7) we could derive

$$\lim_{r \to \infty} \frac{m(r)}{M(r)} > 0 \quad (11)$$

is disproved trivially by the example

$$\sum_{n=0}^{\infty} \frac{z^{n^3}}{(n^3)!} + \sum_{n=0}^{\infty} \frac{z^{n^3+1}}{(n^3+1)!}. \quad (12)$$

Our second generalisation relaxes the gap condition of Theorem 1 in a different way, but imposes in addition a condition on the order of the function. We have

**Theorem 4.**

*If as $n$ tends to infinity*

$$\sum_{k=0}^{n} \frac{1}{\lambda_{k+1} - \lambda_k} = o (\log \lambda_n), \quad (12)$$

*and the function $f(z)$ is of finite order, or if*

$$\sum_{k=0}^{n} \frac{1}{\lambda_{k+1} - \lambda_k} = O (\log \lambda_n), \quad (13)$$

*and $f(z)$ is of zero order, then (2) holds.*

This theorem cannot be materially strengthened since the example
constructed for Theorem 2 will be of finite order if
\[
\lim_{n \to \infty} \frac{1}{\log \lambda_n} \sum_{k=0}^{n} \frac{1}{\lambda_{k+1} - \lambda_k} > 0
\]
and of zero order if
\[
\lim_{n \to \infty} \frac{1}{\log \lambda_n} \sum_{k=0}^{n} \frac{1}{\lambda_{k+1} - \lambda_k} = \infty.
\]

2. Proof of Theorem 1. To prove the theorem we need an elementary inequality. If \( \varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \ldots \) is a convergent series of non-negative numbers and if a sequence \( \delta_n \) is defined by
\[
\delta_n = \max_{i < n < j} \frac{1}{(j - i + 1)^{3/2}} \sum_{v=i}^{j} \varepsilon_v,
\]
then
\[
\sum \delta_n \leq (1 + 2 \sum_{n=2}^{\infty} n^{-3/2}) \sum \varepsilon_n.
\]

We have
\[
\sum \delta_n = \sum_{v=0}^{\infty} A_{v,n} \varepsilon_v,
\]
where \( A_{v,n} = (j_n - i_n + 1)^{-3/2} \) or zero, as \( v \) falls in \( i_n \leq v \leq j_n \) or not, \( i_n, j_n \) being the values of \( i, j \) for which the maximum in (14) is attained. Since \( i_n \leq n \leq j_n \) also it follows that \( j_n - i_n \geq |v - n| \). Consequently
\[
\sum \delta_n \leq \sum_{0}^{\infty} \sum_{0}^{\infty} \frac{\varepsilon_v}{|v - n| + 1}^{3/2}
\]
\[
\leq (1 + 2 \sum_{0}^{\infty} n^{-3/2}) \sum_{0}^{\infty} \varepsilon_n.
\]

We now assume (4) and set
\[
\delta_n = 1/(\lambda_{n+1} - \lambda_n).
\]
Defining \( \delta_n \) as in (14), we have \( \sum \delta_n < \infty \) by (15). Let \( c_n \) be a sequence of positive numbers tending to infinity so slowly that
\[
\sum_{0}^{\infty} c_n \delta_n < \infty.
\]
Now let \( A_n \leq |z| \leq A_{n+1}, n = 0, 1, 2, \ldots, \) be the sequence of intervals in which a single term \( a_k z^k \) remains the maximum term. \( k \) will depend on \( n \) and increases with \( n \), but we need not express this dependence in our notation. From (17) we have \( \prod_{0}^{\infty} (1 + 2c_k \delta_k)^2 < \infty \), and hence there exist arbitrarily large values of \( n \) such that
\[
A_{n+1}/A_n > (1 + 2c_k \delta_k)^2.
\]
We understand by \( n \) such a value and by \( k \) the associated integer. Since \( a_{k} z^{k} \) is the maximum term for \( A_{n} \leq |z| \leq A_{n+1} \), we have

\[
|a_{v}| \leq |a_{k}| A_{n}^{\lambda_{k} - \lambda_{v}} \quad (v < k)
\]

\[
|a_{v}| \leq |a_{k}| A_{n+1}^{-(\lambda_{v} - \lambda_{k})} \quad (v > k).
\]

Using these inequalities with \( r = |z| = (A_{n} A_{n+1})^{1/2} \), we have

\[
|a_{v}| r^{k} \leq |a_{k}| r^{k} (A_{n}/A_{n+1})^{1/2} (\lambda_{k} - \lambda_{v})
\]

\[
\leq |a_{k}| r^{k} (1 + 2 c_{k} \delta_{k}) (\lambda_{k} - \lambda_{v}) \quad (v < k),
\]

\[
|a_{v}| r^{k} \leq |a_{k}| r^{k} (1 + 2 c_{k} \delta_{k}) (\lambda_{v} - \lambda_{k}) \quad (v > k).
\]

But by the definition of \( \delta_{n} \) and the inequality of the harmonic and arithmetic means,

\[
\delta_{k} \geq \left(\frac{1}{\lambda_{v+1} - \lambda_{v}} + \frac{1}{\lambda_{v+2} - \lambda_{v+1}} + \ldots + \frac{1}{\lambda_{k} - \lambda_{k-1}}\right) (k - v)^{-1}
\]

\[
\geq \frac{1}{(k - v)} \left(\frac{k - v}{\lambda_{k} - \lambda_{v}}\right) = \frac{(k - v)^{k}}{\lambda_{k} - \lambda_{v}} \quad (v < k).
\]

Consequently

\[
(1 + 2 c_{k} \gamma_{k}) (\lambda_{k} - \lambda_{v}) \leq e^{-c_{k} (k - v)} \quad (v < k).
\]

From this and a similar inequality when \( v > k \), it follows from (20) that as \( n \to \infty \) (and so \( k \to \infty \), \( r \to \infty \), \( c_{n} \to \infty \))

\[
\sum_{0}^{k-1} |a_{v}| r^{k} + \sum_{k+1}^{\infty} |a_{v}| r^{k} = o\left(|a_{k}| r^{k}\right).
\]

From this follow first the second and then evidently the first statement of (3).

3. **Proof of Theorem 2.** Now suppose that \( \sum_{0}^{\infty} 1/(\lambda_{n+1} - \lambda_{n}) \) diverges. We choose the coefficients \( a_{n} \) by the following rules.

\[
a_{0} = 1, \quad a_{n} = a_{n+1} A_{n}^{-1/(\lambda_{n} - \lambda_{n+1})},
\]

where

\[
A_{n} = \prod_{v=0}^{n-1} \left(1 + \frac{\epsilon_{v}}{\lambda_{v} - \lambda_{v+1}}\right), \quad A_{0} = 1, \quad A_{1} = \left(1 + \frac{1}{\lambda_{0} + 1}\right)
\]

and \( \epsilon_{n} \) is a sequence of positive numbers tending to zero and such that \( \sum \epsilon_{n}/(\lambda_{n+1} - \lambda_{n}) \) diverges.

Evidently \( A_{n} \to \infty \) and \( f(z) = \sum_{0}^{\infty} a_{n} z^{n} \) is an integral function.

Since

\[
a_{n+1} r^{n+1} = \frac{a_{n+1} r^{n+1} - \lambda_{n}}{A_{n+1} - \lambda_{n}},
\]

(26)
the maximum term \( \mu(r) \) is \( a_n r^n \) for

\[
A_n \leq r \leq A_{n+1}.
\]  (27)

Clearly

\[
M(r) = \sum_{0}^{\infty} a_n r^n > a_n r^n + a_{n+1} r^{n+1}.
\]  (28)

Now for \( A_n \leq r \leq A_{n+1} \) we have

\[
a_{n+1} r^{n+1} = \frac{r}{A_{n+1}} \left( \frac{A_{n+1} - A_n}{A_{n+1}} \right) \leq (2 - \epsilon) \mu(r),
\]  (29)

and it follows that \( M(r) > (2 - \epsilon) \mu(r) \) for all sufficiently large \( r \).

This proves the first inequality of (6). To establish the second we argue as follows. With \( A_n \leq r \leq A_{n+1} \) and \( z = re^{\pi/(\lambda_{n+1} - \lambda_n)} \) we have, for \( n \) sufficiently large,

\[
|f(z)| \leq M(r) - a_n r^n - a_{n+1} r^{n+1} + (a_n r^n - a_{n+1} r^{n+1})
\]  (30)

\[
= M(r) - 2 a_{n+1} r^{n+1} \leq M(r) - (2 - \epsilon) \mu(r).
\]

If \( \mu(r) \geq \frac{1}{2} M(r) \), it follows that \( m(r) \leq (\frac{1}{2} + \epsilon) M(r) \).

If \( \mu(r) < \frac{1}{2} M(r) \) we argue differently. We use the relations

\[
\{m(r)\}^2 \leq \{M_2(r)\}^2 = \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_{0}^{\infty} a_n^2 r^{2n},
\]  (31)

which lead to

\[
\{M(r)\}^2 \geq \sum_{0}^{\infty} a_n^2 r^{2n} + \sum_{0}^{\infty} a_n r^n \{f(r) - a_n r^n\}
\]  (32)

\[
\geq \{M_2(r)\}^2 + \sum_{0}^{\infty} a_n r^n \{f(r) - \frac{1}{2} f(r)\}
\]

and

\[
\{m(r)\}^2 \leq \{M_2(r)\}^2 \leq \frac{1}{2} \{M(r)\}^2.
\]  (33)

4. Proof of Theorem 3.

Suppose now that

\[
\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+h} - \lambda_n} < \infty,
\]  (34)

where \( h \) is a positive integer greater than unity.

Defining \( \delta_n \) as in (14) with \( \epsilon_n = (\lambda_{n+h} - \lambda_n)^{-1} \) and choosing \( c_n > 0 \) so that \( c_n \to +\infty \) and \( \sum c_n \delta_n < \infty \), and again taking \( A_n \leq |z| < A_{n+1} \).
to be the sequence of intervals in which a single term, say \(a_k z^{k}\), is the maximum term, we must have arbitrarily large values of \(n\) such that \(A_{n+1}/A_n > (1 + 2c_k^k)^{1/2}\), that is condition (18). With such values of \(n\) and associated \(k\) we still have (19) and (20), but we can no longer expect such a good result as (21) or its consequences (22) and (23). For \(r = \left(An A_{n+1}\right)\) and \(v \approx k\) to \(k\) we can only say

\[
|a_v| r^k \leq |a_k| r^k \quad (k - h < v < k + h) \tag{35}
\]

For values of \(v\) which are not "too near" \(k\) we can give an analogue of (21) valid for \(k - ph < v \leq k - (p - 1)h\), \(\rho = 2, 3, \ldots, p\), in

\[
\delta_k \geq \left(\frac{1}{\lambda_{k-(p-1)h}} + \ldots + \frac{1}{\lambda_{k-h} - \lambda_{k-2h}} + \frac{1}{\lambda_{k} - \lambda_{k-h}}\right) \frac{1}{(ph)^{\rho}}.
\]

Consequently

\[
\left(1 + 2c_k^k\right)^{\delta_k} \leq e^{-c_k^k (x - v)^{1/2}}.
\]

From this and the similar inequalities with \(v > k + h\) we have, as \(n \to \infty\), the result

\[
\sum_{0}^{k-h} |a_v| r^v + \sum_{k+h}^{\infty} |a_v| r^v = o(\sum |a_k| r^{k} ), \tag{36}
\]

and consequently with (35) we deduce

\[
\lim \frac{M(r)}{\mu(r)} \leq (2h - 1)
\]

or

\[
\lim \frac{\mu(r)}{M(r)} \geq 1/(2h - 1),
\]

which constitutes the first part of Theorem 3.

Now suppose that for some integer \(h > 1\)

\[
\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+h} - \lambda_{n}} = \infty.
\]

Then evidently one of the series

\[
\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+h+k} - \lambda_{n+h}} \quad (k = 0, 1, \ldots, h - 1) \tag{37}
\]

must diverge. There will be no loss of generality in supposing that the series with \(k = 0\) diverges. We now, as in the proof of Theorem 2, define the series

\[
f^*(z) = \sum_{n=0}^{\infty} a_n z^{n}, \quad \lambda_n^* = \lambda_{nh}\]
with the properties that
\begin{align*}
(i) \quad \mu^*(r) &= a^*_{n} r^{\lambda_{n}} \\
(ii) \quad a^*_{n+1} r^{\lambda_{n+1}} &\geq (1 - \epsilon) a^*_{n} r^{\lambda_{n}}
\end{align*}
for \( A^*_n \leq r \leq A^*_{n+1}, \quad n > n(\epsilon), \)
where \( \mu^*(r) \) is the maximum term of \( f^*(z) \) and \( A^*_n \) is defined from the sequence \( \lambda^*_n \) as \( A_n \) is defined from \( \lambda_n \) in (25). Let us now define
\( f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n} \) by the conditions
\begin{align*}
a_{nh} &= a^*_{n}, \quad a_{nh+k} = a^*_{n} A^{-\lambda_{nh+h}} \\
(\text{for } k = 1, 2, \ldots, h-1).
\end{align*}

Then evidently for \( A^*_n \leq r \leq A^*_{n+1} \) we shall have
\[ a_{nh} r^{\lambda_{nh}} \geq a_{nh+1} r^{\lambda_{nh+1}} \geq \cdots \geq a_{nh+h} r^{\lambda_{nh+h}}, \]
and \( \mu(r) \) for the function \( f(z) \) will be \( a_{nh} r^{\lambda_{nh}} \), so that
\[ M(r) = f(r) > (h + 1 - \epsilon) \mu(r) \quad [r > r(\epsilon)]. \]

We approximate \( m(r) \) by using
\[ \{m(r)\}^2 \leq \{M_2(r)\}^2 = \sum_{\nu=0}^{\infty} a_{\nu}^2 r^{2\nu}. \]

Clearly
\[ \{M(r)\}^2 = \sum_{\nu=0}^{\infty} a_{\nu}^2 r^{2\nu} + \sum_{\nu=0}^{\infty} a_{\nu} r^{\nu} \{M(r) - a_{\nu} r^{\lambda_{\nu}}\} \]
\[ \geq \{M_2(r)\}^2 + \{M(r)\}^2 - (h + 1 - \epsilon)^{-1} \{M(r)\}^2, \]
from which
\[ m(r) \leq M_2(r) \leq (h + 1 - \epsilon)^{-1} M(r) \]
follows.

This does not quite complete the proof of Theorem 3 since \( (h + 1 - \epsilon)^{-1} \) and \( (h + 1 - \epsilon)^{-1} \), although arbitrarily small, are not zero. However we should only have to choose \( \lambda_n \) to be a subsequence of \( \lambda_n \) such that the interval \( \lambda^*_n \leq \lambda \leq \lambda^*_n+1 \) contains a number of \( \lambda_n \) increasing with \( \lambda^*_n \) but that \( \Sigma (\lambda^*_{n+1} - \lambda^*_n)^{-1} \) diverges. It does not seem necessary to enumerate the details.

5. Proof of Theorem 4.

Given an increasing sequence of integers \( \lambda_n \), let us first try to construct an integral function \( \sum_{n=0}^{\infty} c_n x^{\lambda_n} \) with positive coefficients such that each term is in turn the maximum term and greatly exceeds in
value the rest of the series. More precisely let \( \delta > 0 \) be a small prescribed number and let us choose the \( c_n \) in such a way that for a certain increasing sequence \( A_n \) of positive numbers the following conditions hold for all \( N \). For \( x = A_N \) we require that

\[
c_{N+1} x^{\lambda N+1} = \delta c_N x^{\lambda N}
\]

\[
c_{N-1} x^{\lambda N-1} = \delta c_N x^{\lambda N}.
\]

In this case we shall have, for \( n > N \) and \( x = A_N \),

\[
c_{n+1} x^{\lambda n+1} = \delta c_n x^{\lambda n}
\]

and consequently, for \( x = A_N < A_n \),

\[
c_{n+1} x^{\lambda n+1} \leq \delta c_n x^{\lambda n}.
\]

So for \( x = A_N, \ p > 0 \),

\[
\sum_{N+1}^\infty c_n x^{\lambda n} \leq \frac{\delta}{1-\delta} c_N x^{\lambda N}.
\]

Similarly, for \( x = A_N \),

\[
\sum_0^{N+1} c_n x^{\lambda n} \leq \frac{\delta}{1-\delta} c_N x^{\lambda N}.
\]

We must now consider whether our conditions are possible.

(46) requires that

\[
c_{N+1} = \delta c_N/A_{N+1}^{1-\lambda_N}
\]

\[
c_N = \delta c_{N+1} A_{N+1}^{1-\lambda_N}.
\]

Eliminating \( c_N \) and \( c_{N+1} \), we see that

\[
A_{N+1}/A_N = \delta^{-2/(\lambda_N+1-\lambda_N)} = K^{1/(\lambda_N+1-\lambda_N)} \quad (K > 1).
\]

This defines the sequence \( A_n \) if we take \( A_0 = 1 \), and shows that it is increasing. With \( c_1 = 1 \) the sequence \( c_n \) is also defined, for the two conditions of (46) are now equivalent. The function \( \sum_1^\infty c_n x^{\lambda n} \) will be an integral function if \( A_n \) tends to infinity. Since

\[
\log A_n = \log K \left( \frac{1}{\lambda_1 - \lambda_0} + \frac{1}{\lambda_2 - \lambda_1} + \cdots + \frac{1}{\lambda_n - \lambda_{n-1}} \right),
\]

this condition requires the divergence of \( \sum_1^\infty 1/(\lambda_{n+1} - \lambda_n) \).
The property of domination by single terms expressed by (49) and (50) will be carried over to the integral function \( \sum_0^\infty a_n z^\lambda_n \) if we can assert that
\[
\sum_0^\infty a_n z^\lambda_n / c_n
\]
is an integral function. If we make the hypothesis that \( \sum_0^\infty a_n z^\lambda_n \) is of finite order then \( |a_n| < \lambda_n^{-\alpha z_n} \) for sufficiently large \( n \) and some positive \( \alpha \). To ensure that (54) does define an integral function we shall require to prove that for arbitrary \( \varepsilon > 0 \) and sufficiently large \( n \),
\[
c_n > \lambda_n^{-\varepsilon \lambda_n}.
\]
This is equivalent to
\[
\log c_n > -\varepsilon \lambda_n \log \lambda_n
\]
and since
\[
\log c_n = n \log \delta - \sum_{r=0}^{n-1} (\lambda_{r+1} - \lambda_r) \log A_r
\]
this will follow from
\[
\log A_n = o (\log \lambda_n)
\]
or
\[
\sum_{r=1}^{n} \frac{1}{\lambda_r - \lambda_{r-1}} = o (\log \lambda_n).
\]
Now if we assume that \( \sum_0^\infty a_n z^\lambda_n / c_n \) is an integral function it will follow that for sufficiently large values of \( z \), say \( z = R \), the maximum term of this function will occur with \( n = N \) arbitrarily large. We shall have
\[
|a_n| R^\lambda_n / c_n \leq |a_N| R^\lambda_N / c_N.
\]
Thus the dominance expressed by (49) and (50) of a single term for \( \sum c_n z^\lambda_n \) holds also for the function \( \sum a_n z^\lambda \) with \( |z| = RA_N \). Since \( \delta \) may be chosen arbitrarily small Theorem 4 is proved for functions of finite order. If \( \sum a_n z^\lambda_n \) is assumed to be of zero order we only require that \( c_n > \lambda_n^{-h \lambda_n} \) for some positive \( h \), and this clearly follows from (13).