ON THE PRODUCT OF CONSECUTIVE INTEGERS. III ¹)

BY

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It has been conjectured a long time ago that the product

\[ A_k(n) = n(n+1) \ldots (n+k-1) \]

of \( k \) consecutive integers is never an \( l \)-th power if \( k > 1, l \geq 1 \). Rigge ³) and a few months later I ¹) proved that \( A_k(n) \) is never a square, and later Rigge and I ⁴) proved using the Thue–Siegel theorem that for every \( l \geq 2 \) there exists a \( k_0(l) \) so that for every \( k > k_0(l) \) \( A_k(n) \) is not an \( l \)-th power. In 1940 Siegel and I proved that there is a constant \( c \) so that for \( k > c, l \geq 1 \) \( A_k(n) \) is not an \( l \)-th power, in other words that \( k_0(l) \) is independent of \( l \). Our proof was very similar to that used in ¹) and was never published. A few years ago I obtained a new proof for this result which does not use the result of Thue–Siegel and seems to me to be of sufficient interest to deserve publication. The value of \( c \) could be determined explicitly by a somewhat laborious computation and it probably would turn out to be not too large, and perhaps the proof that the product of consecutive integers is never a power could be furnished by a manageable if long computation (the cases \( k \leq c \) would have to be settled by a different method). A method similar to the one used here was used in a previous paper ⁵).

Now we prove

Theorem 1. There exists a constant \( c \) so that for \( k > c, l \geq 1 \) \( A_k(n) \) is never an \( l \)-th power.

As stated in the introduction Rigge and I proved that \( A_k(n) \) is never a square, thus we can assume \( l \geq 2 \). Further assume that

\[ A_k(n) = x^l. \]

First we need some lemmas.

¹) I had two previous papers by the same title, Journal London Math. Soc. 14, 194–198 (1939) and ibid. 245–249. These papers will be referred to as I and II.

²) A great deal of the early literature of this problem can be found in the paper of R. Oblath, Tohoku Math. Journal 38, 73–92 (1933).

³) O. Rigge, Über ein diophantisches Problem, 9. Congr. des Math. scand. 155–160 (1939) and P. Erdős I.

⁴) P. Erdős, II, As far as I know Rigge proof, which was similar to mine, has not been published.

Lemma 1. $n > k^t$.

First we show $n > k$. If $n < k$ it follows from the theorem of Tchebicheff that there is a prime $p$ satisfying $n \leq \frac{n+k-1}{2} < p \leq n+k-1$. Thus the product $A_k(n)$ is divisible by $p$ but not by $p^2$, or (1) is impossible.

Assume now $n \geq k$. A theorem of Sylvester and Schur \textsuperscript{6}) then asserts that there is a prime $p > k$ which divides $A_k(n)$. But clearly only one of the numbers $n$, $n+1$, ..., $n+k-1$ can be a multiple of $p$, say $n+i = 0 \pmod{p}$. But then we have from (1) $n+i = 0 \pmod{p^2}$ or $n+k-1 \geq n+i \geq (k+1)^2$. Thus $n > k^t$ as stated.

Assume that (1) holds. Since all primes greater or equal to $k$ can occur in at most one term of (1), we must have

$$n+i = a_i x_i^t, \quad 0 \leq i \leq k-1$$

where all the prime factors of $a_i$ are less than $k$ and $a_i$ is not divisible by an $l$-th power.

Lemma 2. The products $a_i a_j$, $0 \leq i, j \leq k-1$, are all different.

Assume $a_i a_j = a_r a_s = A$. Then we would have

$$(n+i)(n+j) = A(x_i x_j), \quad (n+r)(n+s) = A(x_r x_s).$$

First we show that $(n+i)(n+j) = (n+r)(n+s)$ implies $i = r$, $j = s$. Assume first $i+j \neq r+s$, say $i+j > r+s$. Then

$$n^2 + (i+j)n + ij = n^2 + (r+s)n + rs, \text{ or } n \leq rs < k^t$$

which contradicts Lemma 1. Hence $i+j = r+s$, therefore $ij = rs$.

Assume now without loss of generality $(n+r)(n+s) > (n+i)(n+j)$. Then $x_i x_j \leq x_r x_s + 1$ and we would have by Lemma 1

$$2 kn > (n+k-1)^2 - n^2 \geq (n+r)(n+s) - (n+i)(n+j) \geq A [(x_i x_j + 1)^2 - (x_r x_s)^2] > l A (x_i x_j)^{t-1} \geq l [A (x_i x_j)]^{t-1} \geq 3 n^t.$$ 

Thus we would have $n < k^t$, which contradicts Lemma 1. This contradiction proves Lemma 2.

Lemma 3. There exists a sequence $0 \leq i_1 < i_2 < ... < i_t$ so that $t \geq k - \pi(k)$ and

$$(2) \quad \prod_{r=1}^{t} a_{i_r} \mid k!.$$ 

For each $p < k$ denote by $a_{i_p}$ one of the $a_j$'s, $0 \leq j < k$, which have the property that no other $a_j$, $0 \leq j < k$, is divisible by $p$ to a higher power than $a_{i_p}$ (i.e. if $a_j$ is divisible by $p$ to the power $d_j$ then $d_j = \max_{0 \leq j < k} d_j$).

Denote by $a_{i_1}, a_{i_2}, ... a_{i_t}$ the sequence of $a$'s from which all the $a_{i_p}$'s have been omitted. Clearly $t \geq k - \pi(k-1) \geq l - \pi(k)$.

To show that (2) holds it suffices to prove that if \( p^d \) divides the product

\[
\prod_{r=1}^{t} a_r
\]

then \( d \leq [k/p] + [k/p^2] + \ldots \). This is easy to see, since the number of multiples of \( p^d \) among the integers \( n, n+1, \ldots, n+k-1 \) is at most \([k/p^d] + 1\), or the number of multiples of \( p^d \) amongst the \( a_i \)'s, \( 0 \leq i \leq k-1 \), is at most \([k/p^d] + 1\). But then the number of multiples of \( p^d \) among the \( a_i \)'s, \( 1 \leq r \leq t \), is at most \([k/p^d]\), since if there is an \( a_j \equiv 0 \pmod{p^d} \), then \( a_i \equiv 0 \pmod{p^d} \) and \( a_i \) does not occur among the \( a_r \), \( 1 \leq r \leq t \). This completes the proof of the Lemma.

By slightly more complicated arguments we could prove that

\[
\prod_{r=1}^{t} a_r | (k-1)!.
\]

Denote now by \( N(x) \) the maximum number of integers \( 1 \leq b_1 < b_2 < \ldots \\ldots < b_u \leq x \) so that the products \( b_i b_j \), \( 1 \leq i, j \leq u \), are all different.

**Lemma 4.** For sufficiently large \( x \) we have

\[
N(x) < 2x/\log x.
\]

In a previous paper \(^7\) I proved

\[
N(x) < \pi(x) + 8x^{1/4} - x^{1/5}.
\]

Using the well known inequality \( \pi(x) < \frac{3}{2} \frac{x}{\log x} \) we immediately obtain

**Lemma 4.**

For the sake of completeness I will outline a proof of a formula similar to (3) at the end of the paper.

Now we can prove our Theorem. Consider the integers \( a_i \), \( a_u \), \( \ldots, a_t \) of Lemma 3, order them according to size. Thus we obtain the sequence \( b_1 < b_2 < \ldots < b_t \) where by Lemma 2 the numbers \( b_i b_j \) are all different. Let now \( i > i_0 \) be sufficiently large. Putting \( b_i = x \) and using Lemma 4 we obtain

\[
i < N(b_i) < \frac{2b_i}{\log b_i} \quad \text{or} \quad b_i > (i \log i)/2.
\]

Thus from (4) we have for sufficiently large \( i_0 \) and \( t > 2i_0 \)

\[
\prod_{i=1}^{t} b_i > i_0! \prod_{i=i_0+1}^{t} (i \log i)/2 > t! (\log i_0)^{t^2}/t! > t! 10^t.
\]

Now \( t \geq k - \pi(k) > k - \frac{3k}{2 \log k} \). Thus

\[
t! > \frac{k!}{k^{k-1}} > k! k^{-\frac{3k}{2 \log k}} > k! / 5^k.
\]

Thus finally from (5) and (6) we have for sufficiently large $k$

\[(7) \prod_{r=1}^{t} a_r = \prod_{i=1}^{t} b_i > k! \left( \frac{10^t}{5^k} \right) > k! \]

since

\[10^t > 10^{\frac{3k}{\log k}} > 5^k.\]

(7) clearly contradicts Lemma 3, and this contradiction proves the theorem for sufficiently large $k$.

One could easily make the estimations more precise and obtain a better value for $c$, but the method used in this paper does not seem suitable to get a really good value for $c$. The problem clearly is to determine the least constant $c$ so that for all $k > c$ one can not have integers $a_1, a_2, \ldots, a_t$ satisfying (2) $t \geq k - \pi(k)$ and the products $a_i \cdot a_j$ are all distinct.

It is clear from the proof of Theorem 1 that in fact we proved the following slightly stronger result: For $k > c$ there exists a prime $p > k$ so that if $p^\beta \| A(n)$ then $\beta \equiv 0 \pmod{l}$ ($p^\beta \| A(n)$ means: $p^\beta \mid A(n)$, $p^{\beta+1} \nmid A(n)$).

By a slightly more careful estimation at the end of the proof of Theorem 1 we could obtain the following

**Theorem 2.** Let $l > 2$, and $\epsilon$ an arbitrary positive number. Then there exists a constant $c = c(\epsilon)$ so that if $k > c$, $n > k^l$ and we delete from the numbers $n, n + 1, \ldots, n + k - 1$ in an arbitrary way less than $(1 - \epsilon)k\log k/\log k$ of them. Then the product of the remaining numbers is never an $l$-th power.

The condition $n > k^l$ can not entirely be omitted. In fact if $n = 1$ it is easy to see that one can delete $r \leq \pi(k)$ integers from $n, n + 1, \ldots, n + k - 1$ so that the product of the remaining numbers is an $l$-th power.

I can not prove Theorem 2 for $l = 2$, I can only prove it with $c k \log k$ instead of $(1 - \epsilon)k \log k/\log k$.

In the proof of Lemma 3 (1) was not used. Thus if we put

\[ A_i^{[n]} = \prod_{p \mid p^\beta \| n + i, p < k, 0 \leq i \leq k - 1}, \]

we can prove by arguments used in the proof of Lemma 3 that there exists a sequence $i_1, i_2, \ldots, i_t, t > k - \pi(k)$ so that

\[(8) \prod_{r=1}^{t} A_{i_r}^{[n]} \mid (k - 1)!.\]

From (8) it easily follows from the prime number theorem that for $k > k_0 = k_0(\epsilon)$

\[(9) \min_{0 \leq i \leq k - 1} A_i^{[n]} < (1 + \epsilon)k.\]

It is possible that (9) can be sharpened considerably. In fact it is probable that

\[\lim_{k \to \infty} \frac{1}{k} \left( \max_{1 \leq n < \infty} \min_{0 \leq i \leq k - 1} A_i^{[n]} \right) = 0.\]
To complete our proof we now outline the estimation of \( N(x) \). Instead of (3) we shall prove

\[
N(x) < \pi(x) + 3x'^4 + 2x'^6.
\]

It is clear that Lemma 4 is an easy consequence of (10).

Let \( 1 \leq b_1 < b_2 < \ldots < b_s \leq x \) be such that all the products \( b_i b_j, 1 \leq i, j \leq s \), are different. Write \( b_i = u_i v_i \), where \( u_i \) is the greatest divisor of \( b_i \) which is not greater than \( x'^4 \). First of all it is clear that the numbers \( u_1 \cdot v_1, u_1 \cdot v_2, u_2 \cdot v_1, u_2 \cdot v_2 \) can not all be \( b_i \)'s for if \( b_1 = u_1 v_1, b_2 = u_1 v_2, b_3 = u_2 v_1, b_4 = u_2 v_2 \) we would have \( b_1 b_4 = b_2 b_3 \).

Now we distinguish several cases. In case I we have \( u_i < x'^4 \). In this case \( v_i \) must be a prime. For if not let \( p \) be the least prime factor of \( v_i \). If \( p < x'^4 \) then \( pu_i < x'^4 \) which contradicts the maximum property of \( u_i \). Thus \( x'^4 \leq p \leq x'^6 \) (since \( v_i \) was assumed to be composite we evidently have \( p \leq x'^6 \)). But then \( p > u_i \) which again contradicts the maximum property of \( u \). Thus \( v_i \) must be a prime as stated.

Now we distinguish two subcases. In the first subcase are the \( b_i \)'s of the form \( pu_i \), \( u_i < x'^4 \) for which there is no other \( b \) of the form \( pu_i \). The number of these \( b_i \)'s is clearly less than or equal to \( \pi(x) \).

Consider now the \( b_i \)'s of the second subcase. They are clearly of the form

\[
p_i u_i^{(j)}, 1 \leq i \leq r, 1 \leq j \leq l_i, l_i > 1, \ u_i^{(j)} < x'^4.
\]

By what has been previously said each pair of the sets \( U_i, 1 \leq i \leq r \)

\[\{U_i\} = \bigcup_{j} u_j^{(i)}, 1 \leq j \leq l_i\]

can have at most one element in common, or the pairs

\[(u_j^{(1)}, u_j^{(i)}), \ 1 \leq j_1, j_2 \leq l_i, \ 1 \leq i \leq r\]

are all distinct. But since \( u < x'^4 \) the number of these pairs is less than \( x'^4 \).

Thus \( (l_i > 1) \)

\[\sum_{i=1}^{r} \binom{l_i}{2} < x'^4 \quad \text{or} \quad \sum_{i=1}^{r} l_i < 2x'^4.
\]

Hence the number of \( b_i \)'s belonging to the second subcase is less than \( 2x'^6 \).

In the second case \( x'^4 \leq u \leq x'^6 \). Again we consider two subcases. In the first subcase are the \( b_i \)'s of the form \( vu_i \) for which there are at most \( x'^4 \) other \( b_i \)'s of the form \( vu_i \). From \( u_i \geq x'^4 \) we have \( v_i \leq x'^4 \). Thus the number of \( b_i \)'s of the first subcase is clearly less than or equal to \( (x'^4 + 1) \cdot x'^4 \leq 2x'^6 \).

Denote the \( b_i \)'s of the second subcase by

\[v_i u_i^{(i)}, \ 1 \leq i \leq r, \ 1 \leq j \leq l_i, \ l_i > x'^4 + 1.
\]

Again the sets \( U_i, 1 \leq i \leq r \)

\[U_i = \bigcup_{j} u_j^{(i)}, 1 \leq j \leq l_i\]
can have at most one element in common. Thus the pairs \((u_{j_1}^{(i)}, u_{j_2}^{(i)})\), \(1 \leq j_1, j_2 \leq l, 1 \leq i \leq r\) are all distinct. The number of pairs \((u_i, u_n)\) is clearly less than

\[
\left( \left\lfloor \frac{x^{r}}{2} \right\rfloor \right) < \frac{x}{2}.
\]

Thus we have \((l_i > x^{i/r} + 1)\)

\[
\sum_{i=1}^{r} \left( \frac{l_i}{2} \right) < \frac{x}{2} \quad \text{or} \quad \sum_{i=1}^{r} l_i < x^{i/r}.
\]

Thus finally

\[
N(x) < \pi(x) + 3x^{i/r} + 2x^{i/r}
\]

which proves (10).