On Sets Which Are Measured by Multiples of Irrational Numbers

by

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The frequency of naturals \( n \) satisfying a condition \( \Phi \) is defined as the limit

\[
\text{fr}\{n : n \text{ satisfies } \Phi\} = \lim_{N \to \infty} \frac{1}{N} \{n : n \text{ satisfies } \Phi, \ n \leq N\}
\]

provided this limit exists. (\( \bar{A} \) denotes the power of \( A \)).

We say that a set \( A \) (\( A \subset [0, 1] \)) belongs to the class \( \mathcal{E} \) if for every irrational \( \xi \) the frequency \( \text{fr}\{n : n\xi \in A \pmod{1}\} \) exists and does not depend on the choice of \( \xi \). It is well-known that every Jordan measurable set belongs to \( \mathcal{E} \) and, moreover, the frequencies \( \text{fr}\{n : n\xi \in A \pmod{1}\} \) are equal to the measure of \( A \). Further, it is easy to verify that every Hamel base (\( \pmod{1} \)) belongs to \( \mathcal{E} \), which shows that sets belonging to \( \mathcal{E} \) may be Lebesgue non-measurable.

We say that a class \( \mathcal{E}_0 \) is the base of the family \( \mathcal{E} \) if for every \( A \in \mathcal{E} \) there exists a set \( B \in \mathcal{E}_0 \) such that

\[
\text{fr}\{n : n\xi \in A - B \pmod{1}\} = 0
given any irrational \( \xi \).

We say that a class \( \mathcal{E}_1 \) is the weak base of the family \( \mathcal{E} \) if for every \( A \in \mathcal{E} \) there exists a set \( B \in \mathcal{E}_1 \) such that

\[
\text{fr}\{n : n\xi \in A - B \pmod{1}\} = 0
\]

for almost all \( \xi \).

The purpose of this note is the investigation of Lebesgue measurability of sets belonging to a base or to a weak base of the family \( \mathcal{E} \). Namely, we shall prove with the aid of the axiom of choice

\[\star\]

\( A - B \) denotes the symmetric difference of the sets \( A \) and \( B \).
THEOREM 1. Every base of the family $\mathcal{E}$ contains $2^{2^k}$ Lebesgue non-measurable sets.

THEOREM 2. Every weak base of the family $\mathcal{E}$ contains at least $2^{2^k}$ Lebesgue non-measurable sets.

COROLLARY. Under assumption of the continuum hypothesis every weak base of the family $\mathcal{E}$ contains $2^{2^k}$ Lebesgue non-measurable sets.

Before proving the Theorems, we shall prove three Lemmas. Let us introduce the following notations

\begin{equation}
\begin{aligned}
U_k &= \{x : x \text{ rational, } k(k-1) < x < k^2\} \quad (k = 1, 2, \ldots), \\
W_+ &= \bigcup_{k=1}^{\infty} U_k, \quad W_- = \{x : -x \in W_+\}, \quad W = W_+ \cup W_-.
\end{aligned}
\end{equation}

LEMMA 1. For every rational number $r \neq 0$ the equality

\[ fr \{n : nr \in W\} = \frac{1}{2} \]

is true.

Proof. It is sufficient to prove that, for every positive rational number $r$, the equality $fr \{n : nr \in W_+\} = \frac{1}{2}$ holds.

Let $I^{(k)}(r)$ denote the number of such naturals $n$ that $nr \in U_k$.

Obviously,

\[ \left\lfloor \frac{k}{r} \right\rfloor - 1 < I^{(k)}(r) < \left\lfloor \frac{k}{r} \right\rfloor + 1, \]

where $[x]$ denotes the greatest integer $\leq x$.

$I_N(r)$ will denote the number of such naturals $n (n \leq N)$ that $nr \in W_+$.

If $k \leq \sqrt[1]{N_r}$ and $nr \in U_k$ then, in view of (1), $nr < k^2 \leq \sqrt[1]{N_r}^2 < N_r$, which implies the inequality $n < N$. Hence, we obtain the inequality

\[ I_N(r) \geq \sum_{k=1}^{\lfloor \sqrt[1]{N_r} \rfloor} I^{(k)}(r) \quad (N = 1, 2, \ldots). \]

Consequently, taking into account (2), we have the inequality

\[ I_N(r) \geq \sum_{k=1}^{\lfloor \sqrt[1]{N_r} \rfloor} \left\lfloor \frac{k}{r} \right\rfloor - \sqrt[1]{N_r} \quad (N = 1, 2, \ldots). \]

Further, if $k > \sqrt[1]{N_r} + 1$ and $nr \in U_k$ then, in view of (1), $nr \geq k(k-1) > N_r$, which implies the inequality $n > N$. Hence, we obtain the following inequality

\[ I_N(r) \leq \sum_{k=1}^{\lfloor \sqrt[1]{N_r} \rfloor + 1} I^{(k)}(r) \quad (N = 1, 2, \ldots). \]
Consequently, taking into account (2), we have the inequality

\[ I_N(r) \leq \sum_{k=1}^{\lceil \sqrt{N}r \rceil + 1} \left\lfloor \frac{k}{r} \right\rfloor + \lceil \sqrt{N}r \rceil + 1 \quad (N = 1, 2, \ldots). \]

Hence, and from (3), it follows that

\[ I_N(r) = \sum_{k=1}^{\lceil \sqrt{N}r \rceil} \left\lfloor \frac{k}{r} \right\rfloor + o(N) \quad (N = 1, 2, \ldots). \]

Setting \( r = \frac{p}{q} \), \( \lceil \sqrt{N}r \rceil = d_N p + s_N \) \((0 \leq s_N < p)\), where \( p, q, d_N \) and \( s_N \) are integers we obtain by simple reasoning

\[ \sum_{k=1}^{\lceil \sqrt{N}r \rceil} \left\lfloor \frac{k}{r} \right\rfloor = \frac{1}{2} pq d_N (d_N - 1) + d_N \sum_{j=1}^{p} \left\lfloor \frac{j}{r} \right\rfloor + \sum_{j=1}^{s_N} \left\lfloor \frac{j}{r} \right\rfloor + q d_N s_N \]

\[ = \frac{1}{2} pq d_N^2 + o(N) = \frac{1}{2} N + o(N). \]

Hence, in virtue of (4), we obtain the equality \( I_N(r) = \frac{1}{2} N + o(N) \).

The Lemma is thus proved.

By \( \gamma \) we denote the first ordinal number of the power continuum. Let us consider a Hamel base \( x_0 = 1, x_1, x_2, \ldots, x_n, \ldots \) \((a < \gamma)\). Every irrational number \( x \) may be represented as a linear combination with rational coefficients \( x = r_0 + r_1 x_1 + \ldots + r_n x_n \), where \( 1 \leq a_1 < a_2 < \ldots < a_n \), \( r_1 \neq 0 \). In the sequel we shall use the notations \( r(x) = r_1 \), \( a(x) = a_1 \).

Let \( \mathfrak{B} \) be the class of all subsets of the set of all positive ordinals less than \( \gamma \). Obviously,

\[ \mathfrak{B} = 2^{\omega_1}. \]

For every \( V \in \mathfrak{B} \) we define the set

\[ A_V = \{ x: x \text{ irrational, } 0 < x < 1, r(x) \in \mathcal{W}, a(x) \in V \} \cup \{ x: x \text{ irrational, } 0 < x < 1, r(x) \text{ non } \in \mathcal{W}, a(x) \text{ non } \in V \}. \]

**Lemma 2.** For every \( V \in \mathfrak{B} \) \( A_V \in \Xi \). Moreover,

\[ \text{fr} \{ n: n\xi \in A_V(\text{mod}1) \} = \frac{1}{2} \]

for each irrational \( \xi \).

**Proof.** Since \( r(n\xi) = nr(\xi) \) and \( a(n\xi) = a(\xi) \) we have the following equality

\[ \{ n: n\xi \in A_V(\text{mod}1), n \leq N \} = \begin{cases} \{ n: nr(\xi) \in \mathcal{W}, n \leq N \} & \text{if } a(\xi) \in V, \\ \{ n: nr(\xi) \text{ non } \in \mathcal{W}, n \leq N \} & \text{if } a(\xi) \text{ non } \in V. \end{cases} \]
Hence, according to Lemma 1, for every irrational $\xi$, we obtain the equality $\text{fr} \{ n : n\xi \in A_V \pmod{1} \} = \frac{1}{2}$, which was to be proved.

**Lemma 3.** Let $D (D \subset [0, 1))$ be a set satisfying the equality

$$\text{fr} \{ n : n\xi \in A_V - D \pmod{1} \} = 0 \quad (V \in \mathbb{B})$$

for almost all $\xi$. Then, $D$ is Lebesgue non-measurable.

**Proof.** Suppose the contrary, i.e. that $D$ is Lebesgue measurable. First we shall prove that, for every interval $U \ (U \subset [0, 1))$ and for almost all $\xi$,

$$\text{fr} \{ n : n\xi \in A_V \cap U \pmod{1} \} = \frac{1}{2} |U|,$$

where $|U|$ denotes the measure of $U$.

For brevity, we shall use the notations

$$W^0 = W, \quad W^1 = W', \quad V^0 = V \quad \text{and} \quad V^1 = V',$$

where $W'$ denotes the complement of the set $W$ to the set of all rationals and $V'$ denotes the complement of the set $V$ to the set of all positive ordinal numbers less than $\gamma$.

For every rational $r \ (r \neq 0)$ we denote by $k_n^{(i)}(r) \ (n = 1, 2, \ldots)$ the sequence of naturals $n$ such that $nr \in W^i \ (i = 0, 1)$.

It is well-known ([2], p. 344-346) that, for every sequence of integers $k_1 < k_2 < \ldots$ and for every interval $U \ (U \subset [0, 1))$,

$$\text{fr} \{ n : k_n \xi \in U \pmod{1} \} = |U|$$

for almost all $\xi$. Consequently, for almost all $\xi$ and for every rational $r \ (r \neq 0)$, the equality

$$\text{fr} \{ n : k_n^{(i)}(r) \xi \in U \pmod{1} \} = |U| \quad (i = 0, 1).$$

From the definitions of the set $A_V$ and the sequences $k_n^{(i)}(r)$ it follows directly that

$$\{ n : n\xi \in A_V \cap U \pmod{1}, n \leq N \} = \{ n : k_n^{(0)}(r(\xi)) \xi \in U \pmod{1}, k_n^{(0)}(r(\xi)) \leq N \}$$

and

$$\{ n : k_n^{(i)}(r(\xi)) \leq N \} = \{ n : nr(\xi) \in W^i, n \leq N \}$$

if $\alpha(\xi) \in V^i \ (i = 0, 1)$. Hence,

$$\frac{1}{N} \{ n : n\xi \in A_V \cap U \pmod{1}, n \leq N \} =$$

$$= \frac{1}{N} \{ n : nr(\xi) \in W^i, n \leq N \} \frac{\{ n : k_n^{(0)}(r(\xi)) \xi \in U \pmod{1}, k_n^{(0)}(r(\xi)) \leq N \}}{\{ n : k_n^{(0)}(r(\xi)) \leq N \}}$$
if $a(\xi) \in V^i$ $(i = 0, 1)$, which implies, in view of (8) and Lemma 1, the equality
\[
\mathfrak{r}\{n: n\xi \in A_\nu \cap U \pmod{1}\} = \mathfrak{r}\{n: nr(\xi) \in \mathcal{W}^i\} \mathfrak{r}\{n: k_n^{(1)}(r(\xi)) \xi \in U \pmod{1}\} = \frac{1}{2}|U|.
\]

The formula (7) is thus proved.

From (6) and (7) it follows directly that, for every interval $U$ and for almost all $\xi$, the following equality holds
\[
(9) \quad \mathfrak{r}\{n: n\xi \in D \cap U \pmod{1}\} = \frac{1}{2}|U|.
\]

Further, in view of a Theorem of Raikov ([1], p. 377),
\[
\lim_{N \to \infty} \int_0^1 \left| \sum_{n=1}^{N} \chi(n\xi) - |D \cap U| \right| d\xi = 0,
\]
where $\chi$ is the characteristic function of $D \cap U$ extended on the line with the period 1. Hence, and from (9), for every interval $U$, we obtain the equality $|D \cap U| = \frac{1}{2}|U|$, which contradicts the Lebesgue density theorem. The Lemma is thus proved.

**Proof of Theorem 1.** Let $V \in \mathfrak{B}$. By $B_V$ we denote a set belonging to the base of the family $\mathcal{E}$ such that
\[
\mathfrak{r}\{n: n\xi \in A_V \cap B_V \pmod{1}\} = 0
\]
for each irrational $\xi$. (According to Lemma 2 the sets $A_V$ ($V \in \mathfrak{B}$) belong to $\mathcal{E}$). Applying Lemma 3 we find that the sets $B_V$ are Lebesgue non-measurable. Since the power of the base is $\leq 2^{2^\mathfrak{b}}$, then, to prove the Theorem, it suffices to show, in virtue of (5), that the function $V \mapsto B_V$ establishes a one-to-one correspondence between sets $V$ and sets $B_V$.

Suppose $V_1 \neq V_2$. There is then an irrational $\xi_0$ such that $a(\xi_0) \in V_1 \cap V_2$. Taking into account the definition of $A_V$, we have $n\xi_0 \in A_{V_1} \cap A_{V_2} \pmod{1}$ ($n = 1, 2, \ldots$). Hence, $\mathfrak{r}\{n: n\xi_0 \in A_{V_1} \cap A_{V_2} \pmod{1}\} = 1$, which implies $\mathfrak{r}\{n: n\xi_0 \in B_{V_1} \cap B_{V_2} \pmod{1}\} = 1$. Consequently, $B_{V_1} \neq B_{V_2}$.

Theorem 1 is thus proved.

**Proof of Theorem 2.** By $\mathfrak{B}_0$ we denote the class of all subsets of the set of all denumerable ordinal numbers. Obviously, $\mathfrak{B}_0 \subset \mathfrak{B}$ and $\mathfrak{B}_0 = 2^{\mathfrak{b}}$. By $C_V$ ($V \in \mathfrak{B}_0$) we denote a set belonging to the weak base of the family $\mathcal{E}$ such that
\[
\mathfrak{r}\{n: n\xi \in A_V \cap C_V \pmod{1}\} = 0
\]
for almost all $\xi$. According to Lemma 3, the sets $C_V$ are Lebesgue non-measurable. To prove the Theorem it suffices to show that the function: $V \mapsto C_V$ ($V \in \mathfrak{B}_0$) establishes a one-to-one correspondence between the
sets $V$ and the sets $C_V$. Suppose $V_1 \neq V_2$ ($V_1, V_2 \in \mathcal{B}_0$). Similarly to the preceding proof we find that

$$\text{fr}\{n: n\xi \in C_{V_1} \cap C_{V_2} \pmod{1}\} = 1$$

for almost all $\xi$ satisfying the condition $a(\xi) \epsilon V_1 \cap V_2$. Obviously, to prove the inequality $C_{V_1} \neq C_{V_2}$ it is sufficient to show that the outer Lebesgue measure of the set $S = \{\xi: a(\xi) \epsilon V_1 \cap V_2\}$ is positive. Suppose the contrary, i.e.

$$|S| = 0.$$  

Let $\eta$ be the first ordinal number belonging to $V_1 \cap V_2$. It is easy to verify that the real line $R$ may be represented as the denumerable union of sets congruent to $S$

$$R = \bigcup_{r_1, \ldots, r_n} \left\{ x + \sum_{i=1}^{n} r_i x_{r_i}: x \epsilon S \right\},$$

where $r_1, \ldots, r_n$ are rationals ($n = 1, 2, \ldots$). Hence, and from (10), it follows that $|R| = 0$, which is impossible. The Theorem is thus proved.

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REFERENCES
