THE TOPOLOGIZATION OF A SEQUENCE SPACE BY TOEPLITZ MATRICES

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1. INTRODUCTION

It is our purpose to show how the convergence fields of certain regular Toeplitz matrices can serve as neighborhoods, in the topologization of a factor space of the space m of bounded sequences. We use two simple ideas: the principle of aping sequences, and the construction of Toeplitz matrices whose convergence fields are nontrivial but (in the sense in which this is possible) arbitrarily small.

The basic idea of the principle of aping sequences was introduced independently by Agnew [1, Theorem 4.1] and by Brundno [2]. But Agnew announced the principle only in a specialized form; and Brundno, who showed great skill in developing elaborate and highly fruitful applications, failed to make the principle explicit. Roughly, the idea can be stated as follows: *If a Toeplitz matrix* A *is regular* (or, more generally, if it has finite norm and if all its column limits exist), *then it transforms each pair of fairly similar bounded sequences into a pair of fairly similar bounded sequences.* Naturally, the expression "fairly similar" must in this context be defined in terms of the matrix A.

Terminology and Notation. By $A = (a_{nk})$ and $B = (b_{nk})$ we denote either Toeplitz matrices or the transformations which they represent. By a sequence

$$x = \{x_k\} = \{x(k)\}$$

we mean a sequence of complex constants, and by Ax the sequence $\{\sum_{k=0}^{\infty} a_{nk} x_k\}_{n=0}^{\infty}$, that is, the transform of x by A. Our matrices are regular, and the symbol Ax will never be used except where it is meaningful. If the sequence Ax converges to the number a, we write $Ax \to a$, and we say that A *evaluates* x *to* a.

The statement $x \sim y$ will mean that there exists a nonzero constant λ and a convergent sequence c such that $y_k = \lambda x_k + c_k$. Clearly, the relation $y \sim x$ is an equivalence relation; we denote by m/L the space of equivalence classes which it determines in m, and we use the letter X to represent the equivalence class to which x belongs. If $x \sim y$, then $Ax \sim Ay$, for every regular matrix A; we therefore use the symbol AX to denote the equivalence class to which Ax belongs. The set of bounded sequences evaluated by A is called the *convergence field of* A *in* m (or the *bounded convergence field of* A). By an immediate extension, we can speak of the convergence field of A in m/L.

If $\{k_r\}$ is an increasing sequence of positive integers, we say that $\{\xi_k\}$ *wanders slowly over* $\{k_r\}$ *provided*

$$\lim_{r\to\infty} \max_{k_r < k \leq k_{r+1}} |\xi(k) - \xi(k_r)| = 0.$$

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If ξ wanders slowly over $\{k_r\}$ and η wanders slowly over $\{n_r\}$, and if

$$\xi(k_r) - \eta(n_r) \to 0,$$

we say that ξ and η are $\{k_r, n_r\}$-*similar*. We say that y *apes* x *(boundedly) over* $\{k_r\}$ provided there exists a (bounded) sequence ξ wandering slowly over $\{k_r\}$ and having the property that $\{y_k - x_k \xi_k\}$ converges. The sequence ξ is called an *aping factor* of y, relative to x. (For an extensive development of related ideas, see Zeller [13].)

2. THE PRINCIPLE OF APING SEQUENCES

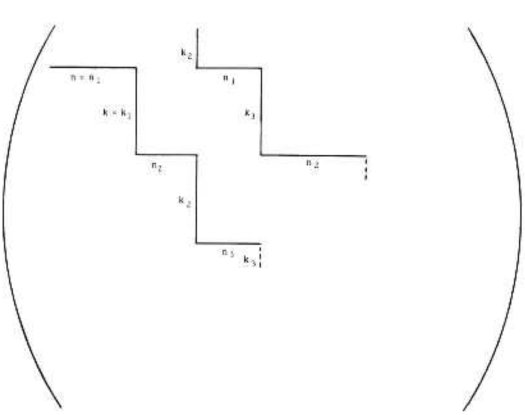

Figure 1.

Whenever the discussion is restricted to regular Toeplitz matrices and bounded sequences, it may be assumed, for all our purposes, that the matrices are row-finite. More generally, it may be assumed that the nonzero elements of each matrix lie in rectangular blocks arranged in such a way that no column and no row meets more than two of the blocks (see Figure 1). To make this clear, let A be a regular matrix; let $k_0 = 0 = n_0$, and let k_1 be any positive integer. An elementary step-by-step process yields integers $n_1, k_2, n_2, k_3, n_3, \cdots$ such that $\{k_r\}$ and $\{n_r\}$ are increasing sequences and

$$\lim_{r\to\infty} \max_{n_r \le n < n_{r+1}} \left(\sum_{k<k_r} + \sum_{k>k_{r+2}} \right) |a_{nk}| = 0 \tag{1}$$

(see also [3, Theorem 3]. We call the set of elements a_{nk} that occur in the right member of (1) the $\{k_r, n_r\}$-*trim of* A, and we paraphrase the relation (1) by saying that the $\{k_r, n_r\}$-trim of A tends to 0. Clearly, if A′ is obtained by replacing by 0 each element that occurs in the $\{k_r, n_r\}$-trim of A, then $(A - A')x \to 0$, for each $x \in m$.

THEOREM 1 (Principle of Aping Sequences). *Let* A *be a regular matrix, and let* $\{k_r, n_r\}$ *be a sequence of index pairs such that the* $\{k_r, n_r\}$-*trim of* A *tends to* 0. *Let* $x \in m$, *and let* $y_k = x_k \xi_k + c_k$, *where* ξ *is bounded and wanders slowly over* $\{k_r\}$, *and where* $c_k \to \alpha$, *and let* $s = Ax$, $t = Ay$. *Then* t *has a representation* $t_n = s_n \eta_n + \gamma_n$, *where* $\gamma_n \to \alpha$, *where the sequence* η *wanders slowly over* $\{n_r\}$, *and where* ξ *and* η *are* $\{k_r, n_r\}$-*similar*.

To prove the theorem, we write, for $n_r \le n < n_{r+1}$,

$$\begin{aligned} t_n = \sum_{k<k_r} a_{nk} y_k + \xi(k_r) \sum_{k_r \le k \le k_{r+2}} a_{nk} x_k + \sum_{k_r \le k \le k_{r+2}} a_{nk}[\xi(k) - \xi(k_r)] x_k \\ + \sum_{k_r \le k \le k_{r+2}} a_{nk} c_k + \sum_{k>k_r} a_{nk} y_k = S_1 + S_2 + S_3 + S_4 + S_5. \end{aligned} \tag{2}$$

As $n \to \infty$, S_1 and S_5 tend to zero, since y is bounded and (1) is satisfied; together with the regularity of A, condition (1) implies further that $S_2 - \xi(k_r)s_n \to 0$ and that $S_4 \to \alpha$; the hypothesis on ξ and the boundedness of x imply that $S_3 \to 0$. This completes the proof. A special case of the theorem was proved by Agnew [1, p. 98].

By way of an immediate application, we give a brief proof of Brudno's famous theorem: *If* A *and* B *are regular Toeplitz matrices such that the bounded convergence field of* A *includes that of* B, *then* A *and* B *are consistent in the space* m. (See [2, p. 198]; see also Mazur and Orlicz [7, Theorem 6] and [8], and Zeller [13, Theorem 6.4].)

There exist increasing sequences $\{k_r\}$ and $\{n_r\}$ such that the $\{k_r, n_r\}$-trims of A and B both tend to 0. If A and B are not consistent in m, there exists an $x \in m$ such that $Ax \to 1$ and $Bx \to 0$. Now let ξ be a bounded sequence wandering slowly over $\{k_r\}$, with $\xi(k_r) = \sin\sqrt{r}$; then B evaluates the sequence $\{x_k\xi_k\}$ to 0, while A transforms $\{x_k\xi_k\}$ into a sequence which oscillates slowly between 1 and -1, contrary to the hypothesis that $A \supset B$ in m. We point out that this is essentially Brudno's proof [2, pp. 198-205], also the proof given by Petersen [9].

Theorem 1 can be modified so that it yields the following theorem of Darevsky [4] (see also Copping [3, Theorem 5] and Zeller [12, Theorem 7.1], [13, Theorem 9.1]): *If* A *is a regular Toeplitz matrix which evaluates a divergent sequence, then* A *evaluates an unbounded sequence.*

To see this, suppose that x is a bounded divergent sequence, and let A denote a regular Toeplitz matrix which evaluates x to 0. A decomposition of the sum $\Sigma a_{nk}x_k\xi_k$ analogous to (2) shows that if $\xi_k \to \infty$ slowly enough, then A evaluates the unbounded sequence $\{x_k\xi_k\}$ to 0.

3. THE CONSTRUCTION OF SMALL CONVERGENCE FIELDS

THEOREM 2. *Let* x *and* a *denote a bounded divergent sequence and a complex number, respectively, and let* $\{k_r\}$ *be an increasing sequence of integers such that* $x(k_{2r}) \to \alpha$ *and* $x(k_{2r+1}) \to \beta$ $(\alpha \neq a, \beta \neq a, \alpha \neq \beta)$. *Then there exists a regular Toeplitz matrix* A *such that*

(i) $$Ax \to a,$$

(ii) Ay *converges if and only if* y *apes* $\{x_k - a\}$ *over* $\{k_r\}$,

(iii) $$\inf |a_{nn}| > 0, \textit{ and } a_{nk} = 0 \textit{ whenever } k > n.$$

Condition (iii) can be paraphrased by saying that A is "strongly normal." Together with the regularity of A, it implies that A has a unique inverse which is also strongly normal.

In the proof of the theorem, we deal with the special case where $a = 0$; the general case can be reduced to this by dealing with the sequence $\{x'_k\} = \{x_k - a\}$. We also suppose that, for all r,

$$|x(k_{2r}) - \alpha| < \gamma, \qquad |x(k_{2r+1}) - \beta| < \gamma,$$

where $\gamma = |\alpha - \beta|/4$; the condition is always satisfied when r is large enough, and the exceptional cases can be removed by deleting the corresponding elements from $\{k_r\}$. Similarly, it may be assumed that $|x(k_{2r})| > |\alpha|/2$ for $r > 1$.

For $k \leq k_2$, let the kth row of A be the kth row of the identity matrix. Corresponding to each $k > k_2$, let p (more precisely, p_k) denote the greatest of the integers k_{2r} ($k_{2r} < k$) or k_{2r+1} ($k_{2r+1} < k$), according as the inequality $|x_k - \alpha| > |x_k - \beta|$ does or does not hold. Let the kth row of A be defined by the rule

$$a_{kp} = x_k/(x_k - x_p), \quad a_{kk} = -x_p/(x_k - x_p), \quad a_{kh} = 0 \quad (h \neq p, k).$$

Since x is bounded and $|x_k - x_p| > \gamma$ for all $k > k_2$, the matrix A is regular and satisfies condition (iii). Inspection shows that (i) holds.

To establish (ii), suppose first that $y_k = x_k \xi_k + c_k$, where $\{c_k\}$ converges, and let $Ay = t$. Then, with $p = p_n$,

$$t_n = [x_n(x_p\xi_p + c_p) - x_p(x_n\xi_n + c_n)]/(x_n - x_p)$$
$$= [x_n c_p - x_p c_n + x_n x_p(\xi_p - \xi_n)]/(x_n - x_p);$$

this shows that A evaluates every bounded or unbounded sequence y which apes $\{x_k - a\}$ over $\{k_r\}$.

Suppose, on the other hand, that $t = Ay \to 0$ (if $Ay \to b \neq 0$, we can deal with the sequence $\{y_k - b\}$). We shall construct a sequence ξ such that y apes x over $\{k_r\}$ with the aping factor ξ. By hypothesis, $t(k_r) \to 0$, that is,

$$x(k_r)y(k_{r-1}) - x(k_{r-1})y(k_r) \to 0.$$

Since the sequence $\{x(k_r)\}$ is bounded away from zero, this suggests the choice $\xi(k_r) = y(k_r)/x(k_r)$, for $r = 3, 4, \cdots$, and it remains only to supply the remaining elements of $\{\xi_k\}$. We now write $\xi_k = y_p/x_p$ ($k \neq k_1, k_2, \cdots$; $p = p_k$); then, from the relation

$$t_n = x_p(x_n\xi_n - y_n)/(x_n - x_p) \qquad (p = p_n)$$

and from the hypothesis that $t_n \to 0$, it follows that $x_k\xi_k - y_k \to 0$. Since each p_n is one of the two largest indices k_r which are less than n, ξ wanders slowly over $\{k_r\}$, and the proof is complete.

THEOREM 2'. *If the hypothesis* $\alpha \neq$ a *is removed from Theorem* 2, *the theorem remains valid provided* (ii) *is replaced by the weaker statement*

(ii') *each sequence evaluated by* A *apes* $\{x_k - a\}$ *over* $\{k_r\}$, *and* A *evaluates each bounded sequence that apes* $\{x_k - a\}$ *over* $\{k_r\}$.

In the proof, it may be assumed that $\beta = a = 0$ and $|x(k_{2r})| > |\alpha|/2$; for each case can either be reduced to this, or it is covered by the proof of Theorem 2. If k is one of the integers k_{2r+1}, we define the kth row of A by the rule

$$a_{kk} = 1, \quad a_{kh} = 0 \quad (h \neq k).$$

For $r = 2, 3, \cdots$ and $k_{2r-1} < k < k_{2r+1}$, we define

$$a(k, k_{2r-2}) = x(k)/x(k_{2r-2}), \quad a(k, k_{2r-1}) = -x(k)/x(k_{2r-2}), \quad a(k, k) = 1,$$
$$a(k, h) = 0 \quad (h \neq k_{2r-2}, k_{2r-1}, k).$$

The remaining details of the proof are mechanical, and we omit them.

THEOREM 2". *Let* x *and* a *denote a bounded divergent sequence and a complex number, respectively; and let* $\{k_r\}$ *be an increasing sequence such that* $x(k_{2r}) \to \alpha$ *and* $x(k_{2r+1}) \to \beta$ $(\alpha \neq \beta)$. *If* $x(k_{2r+1}) - \beta = O(1/r)$, *there exists a regular Toeplitz matrix* A *with the properties* (i), (ii), (iii) *in Theorem* 2.

The theorem can be proved by means of the matrix which has just been described. In proving the second part of (ii), we use the following fact: If $y_k = x_k \xi_k$, where ξ wanders slowly over $\{k_r\}$, then the hypothesis on $\{x(k_{2r+1})\}$ implies that $y(k_{2r+1}) = o(1)$.

To facilitate the comparison of Theorems 2, 2' and 2", we point out the following differences: Theorem 2 requires that the value a to which the sequence Ax is to converge be different from each of the two limit points α and β of x. Theorem 2" replaces this condition by the requirement that $x(k_{2r+1}) \to \beta$ fairly rapidly. Theorem 2' simply omits the condition; but it fails to guarantee that A evaluates all unbounded sequences that ape x over $\{k_r\}$.

We now turn our attention to the case where the sequence x converges. If $x_k \to 0$, and if $k_r \to \infty$ sufficiently fast, then every sequence which apes x over $\{k_r\}$ is convergent. Therefore the case of nullsequences carries little interest. All other cases can be covered by dealing with the sequence $x = \{1\} = \{1, 1, \cdots\}$.

THEOREM 3. *If* $k_r \to \infty$, *there exists a regular row-finite Toeplitz matrix which evaluates some bounded divergent sequences, and which evaluates no sequence that does not ape* $\{1\}$ *over* $\{k_r\}$.

The theorem can be proved by means of a construction much in the spirit of what has gone before. We shall not discuss the details. Instead, we point out why Theorem 3 can not serve our purpose of providing neighborhoods of the equivalence class C of convergent sequences: if A is a regular Toeplitz matrix and if $h_r \to \infty$ fast enough, then (by the principle of aping sequences) A evaluates no bounded divergent sequence which apes $\{1\}$ over $\{h_r\}$.

4. A PROBLEM OF STEINHAUS TYPE

If A is a regular Toeplitz matrix, if the sequence x consists of alternate blocks of 0's and 1's of lengths $L_1, L_2, \cdots$, and if $L_r \to \infty$ rapidly enough, then Ax does not converge [11]. In a private communication, Agnew asked whether there exists a regular Toeplitz matrix that evaluates some bounded divergent sequences but evaluates no divergent sequences of 0's and 1's. The preceding section clearly gives an affirmative answer, as does also the matrix Z_3 of Silverman and Szasz [10, p. 347]; but the following theorem makes an even stronger assertion.

THEOREM 4. *There exists a regular Toeplitz matrix* A *which evaluates some bounded divergent sequences and has the following property: If* $Ay \to 0$, *then the set of limit points of* y *is either the origin, or a circular disk centered at the origin, or the entire plane.*

To prove this, let x be a sequence whose set of limit points is the unit disk, and let $\{k_r\}$ be a sequence of integers such that each point in the unit disk lies at a distance less than $1/r^2$ from the point set $\{x_k\}$ $(k_r < k < k_{r+1})$. By Theorem 2, there exists a regular Toeplitz matrix A such that $Ay \to 0$ if and only if $y_k = x_k \xi_k$, where ξ wanders slowly over $\{k_r\}$. The remainder of the proof is tedious, but trivial, and we omit it.

5. THE TOPOLOGIZATION OF $m/L - \{C\}$

If A is a regular Toeplitz matrix which evaluates at least one bounded divergent sequence, then the convergence field of A in m/L contains nondenumerably many points. On the other hand, if x and y are two bounded divergent sequences such that every regular matrix that evaluates x also evaluates y, then $X = Y$ (by Theorem 2; see also Erdös and Rosenbloom [5]). This suggests the possibility of using convergence fields as neighborhoods, in the topologization of m/L.

From the remarks at the end of Section 3 it is clear that, in any topologization of m/L by means of convergence fields of regular Toeplitz matrices, the equivalence class C of convergent sequences emerges as a discrete point. We shall therefore deal with the space $m/L - \{C\}$.

The following theorem implies that if the convergence field in $m/L - \{C\}$ of each regular Toeplitz matrix is admitted as a neighborhood of each of its points, then every point in $m/L - \{C\}$ is a discrete point.

THEOREM 5. *If* x *is a bounded divergent sequence, if the elements of the sequence* $\{k_r\}$ *are sufficiently far apart, and if* a *and* b *are two constants* $(a \neq b)$, *then no divergent sequence* y $(Y \neq X)$ *apes both* $\{x_k - a\}$ *and* $\{x_k - b\}$ *over* $\{k_r\}$.

To prove this theorem, let α and β be two distinct limit points of x, and suppose that $k_r \nearrow \infty$, $x(k_{2r}) \to \alpha$, $x(k_{2r+1}) \to \beta$. Suppose that

$$y_k = (x_k - a)\xi_k + p_k = (x_k - b)\eta_k + q_k,$$

where $a \neq b$, $p_k \to p$, $q_k \to q$, and where ξ and η wander slowly over $\{k_r\}$; clearly, both ξ and η are bounded if and only if y is bounded.

Suppose first that y is bounded, and let $\{h_s\}$ be a subsequence of $\{k_r\}$ such that $\{\xi(h_s)\}$ and $\{\eta(h_s)\}$ converge to limits u and v, respectively. We may assume that when $k_r = h_s$, then r and s have the same parity. This gives the relations

$$\lim_{s\to\infty} y(h_{2s}) = (\alpha - a)u + p = (\alpha - b)v + q,$$

$$\lim_{s\to\infty} y(h_{2s+1}) = (\beta - a)u + p = (\beta - b)v + q,$$

from which we deduce, successively, that

$$(\alpha - \beta)u = (\alpha - \beta)v, \qquad u = v, \qquad u = (p - q)/(a - b).$$

The last equation shows that ξ and η can not have any limit point other than $(p - q)/(a - b)$, and therefore that $x \sim y$.

If y is unbounded, there exists a sequence $\{h_s\}$ such that $\xi(h_s) \to \infty$ and $\eta(h_s) \to \infty$. Clearly, $\xi(h_s)/\eta(h_s) \to 1$, and it follows that

$$(\alpha - a)(\beta - b) = (\alpha - b)(\beta - a),$$

in other words, $(a - b)(\alpha - \beta) = 0$. This completes the proof.

Theorem 5 implies that if the space $m/L - \{C\}$ is to have no discrete points, then the set of Toeplitz matrices whose convergence fields in $m/L - \{C\}$ are admitted as neighborhoods must somehow be restricted. On the other hand, the principle of aping sequences entails the following proposition.

THEOREM 6. *If* S *is a system of mutually consistent regular Toeplitz matrices such that each bounded sequence is evaluated by some* A *in* S, *and such that to each pair of distinct points* X *and* Y *in* $m/L - \{C\}$ *there correspond two matrices* A *and* B *in* S *whose convergence fields in* $m/L - \{C\}$ *are disjoint and contain* X *and* Y, *respectively; then the system* S *can be enlarged to a consistent system* S* *whose convergence fields determine a Hausdorff topology without discrete points, in* $m/L - \{C\}$.

That each point X in $m/L - \{C\}$ has at least one neighborhood U(X) follows directly from the first part of our hypothesis. If U(X) and V(X) are any two neighborhoods of X, determined by matrices A and B, we form the matrix whose (2n)th and (2n + 1)st rows are the nth rows of A and B, respectively $(n = 0, 1, \cdots)$; then this matrix determines the neighborhood $W(X) = U(X) \cap V(X)$. That every neighborhood is open is a tautology. Finally, the separation axiom is the substance of the second part of our hypothesis. We shall now use Theorem 6 to give two proofs of the following proposition.

THEOREM 7. *There exists a system* S *of regular Toeplitz matrices whose bounded convergence fields determine a Hausdorff topology without discrete points, in* $m/L - \{C\}$.

Our first proof will be very direct; but it will allow no choice in the topology of the space. The second proof will be based on an unproved set-theoretic hypothesis; but it will permit a free choice, at each of $2^{\aleph_0}$ stages, in the sense that the value to which a sequence x shall be evaluated by the matrices in S which evaluate it can be chosen arbitrarily (in fact, the proof can be arranged in such a way that the free choice applies to each of $2^{\aleph_0}$ sequences of 0's and 1's).

The first proof. For any bounded divergent sequence x, let K(x) denote the core of x, that is, the least convex set (in the Cartesian plane) which contains all limit points of x. There exists a unique circular disk of minimum radius which contains K(x) in its closure. We call the center p(x) of this disk the *center of the sequence.*

For each x $(X \in m/L - \{C\})$, there exists a sequence $\{k_r\}$ such that if ξ wanders slowly and boundedly over $\{k_r\}$, then the sequence $\{y_k\} = \{[x_k - p(x)]\xi_k\}$ has the center p(x). By Theorem 2, we can construct a system S of regular Toeplitz matrices, each with the property that if it evaluates a bounded divergent sequence z, it evaluates it to p(z); moreover, the system can be made both extensive and refined enough so that it satisfies the hypothesis of Theorem 6. (For an example of a system of mutually consistent matrices whose collective convergence fields cover the space of real, bounded sequences, see Goffman and Petersen [6].)

The second proof. We shall need the following lemma, which has some independent interest.

LEMMA. *If* A *is a regular row-finite Toeplitz matrix which does not evaluate the bounded sequence* x, *then there exists a sequence* $\{k_r\}$ *such that* A *evaluates no divergent sequence which apes* x *over* $\{k_r\}$.

Let Ax = t, and let $\{m_s\}$ be an increasing sequence of integers such that $t(m_{2s}) \to \alpha$, $t(m_{2s+1}) \to \beta$ $(\alpha \neq \beta)$. There exists a sequence $\{k_r, n_r\}$ of index pairs such that $\{n_r\}$ is a subsequence of $\{m_s\}$, such that the $\{k_r, n_r\}$-trim of A tends to 0 so rapidly that the quantity under the limit sign in equation (1) is $O(1/r)$, and such that $t(n_{2r}) \to \alpha$ and $t(n_{2r+1}) \to \beta$. Now let y be any sequence such that

$y_k = x_k \xi_k$, where ξ wanders slowly over $\{k_r\}$. Then Ay can not converge unless $\xi_k \to 0$.

To return to the proof of Theorem 7, suppose that the set of points in $m/L - \{C\}$ has been well-ordered into a transfinite sequence $\{X_\alpha\}$ $(0 < \alpha < \omega$, where ω denotes the first ordinal of cardinality $2^{\aleph_0})$. Corresponding to X_1, we choose any matrix A_1 which evaluates X_1 (if we wish, we can think of the choice as being made mechanically by some well-ordering mechanism).

Suppose now that regular matrices A_α have been chosen for $0 < \alpha < \beta$, and that their convergence fields in $m/L - \{C\}$ are disjoint. Also, let X denote the first point in $m/L - \{C\}$ which is not contained in the convergence field of any of these matrices; let $x \in X$, and let p denote any complex number. By our lemma there exists, for each $\alpha < \beta$, a sequence $\{k_r(\alpha)\}_{r=1}^{\infty}$ such that no divergent sequence aping $\{x_k - p\}$ over $\{k_r(\alpha)\}$ is evaluated by A_α. We now state our unproved set-theoretic assumption; it is clearly a consequence of the continuum hypothesis.

HYPOTHESIS. *Let* K *denote a family of fewer than* $2^{\aleph_0}$ *increasing sequences* $\{k_r\}$. *Then there exists a sequence* $\{h_s\}$ *with the following property: if* $\{k_r\} \in K$, *then at most finitely many intervals* $h_s < h \leq h_{s+1}$ *contain more than one of the numbers* k_r.

Under the hypothesis, Theorem 2′ permits the construction of a matrix A_β whose convergence field contains the point X but meets the convergence field of none of the matrices A_α $(\alpha < \beta)$. Therefore, we can construct a system $S = \{A_\beta\}$ $(0 < \beta < \omega)$ of disjoint convergence fields which cover the space $m/L - \{C\}$. By Theorems 2′ and 6, the system can be extended so that it has the property stated in Theorem 7.

6. SOME PROPERTIES OF $m/L - \{C\}$.

We shall now assume that the space $m/L - \{C\}$ has been topologized by a system S such as was constructed in the preceding section, and we shall exhibit some topological properties of the space.

THEOREM 8. *If* A *is a regular Toeplitz matrix, its convergence field in* $m/L - \{C\}$ *is a closed set.*

COROLLARY. *The neighborhoods in* $m/L - \{C\}$ *have no boundary points.*

The theorem follows immediately from the lemma in Section 5.

THEOREM 9. *If* E *is a neighborhood of a point* X *in* $m/L - \{C\}$, *there exists a family* $\{E_f\}$ *of neighborhoods of* X (*where the index* f *ranges over all real numbers in* $[0, 1]$) *such that* $E_f \subset E_g \subset E$ *whenever* $0 \leq f \leq g \leq 1$, *and such that each* E_f *contains an open set which meets none of the sets* E_g $(g < f)$.

For an intuitive picture of the family $\{E_f\}$, one may think of a nested system of $2^{\aleph_0}$ disks D_α with the following property: each disk D_α contains an open set which meets none of the disks D_β $(\beta < \alpha)$.

To prove the theorem, let $\{k_r\}$ be a sequence determining a neighborhood of X which is contained in E. First we construct a family of subsequences $\{k_{fr}\}$ of $\{k_r\}$ with the following property: if $g < f$, then $\{k_{gr}\}$ is a subsequence of $\{k_{fr}\}$, and $\{k_{fr}\}$ contains arbitrarily long blocks of elements that do not occur in $\{k_{gr}\}$.

Let each number f in $(0, 1]$ be represented by a dyadic symbol $0.a_1a_2\cdots$, where infinitely many of the a_i are equal to 1. Let the sequence $\{k_r\}$ be divided into consecutive sections of 1^2, 2^4, 3^8, 4^{16}, $\cdots$ elements, respectively. The sequence $\{k_{fr}\}$ is then built, section by section, the elements of its pth section being taken from the pth section of $\{k_r\}$.

We regard the pth section of $\{k_r\}$ as its own subsection of first rank. We divide it into p equal sections, called the subsections of second rank; this process is continued until we arrive at p^{2^k} subsections of rank 2^k; each of these consists of p elements.

The terminating dyadic fraction $f_p = 0.a_1 \cdots a_p$ represents one of 2^p possible numbers. If f_p has the least possible value of a dyadic fraction of p digits, we declare that the pth section of $\{k_{fr}\}$ shall consist of the first element of the pth subsection of rank 1; more generally, if f_p has the jth of the 2^p possible values, then the pth section of $\{k_{fr}\}$ shall consist of the p^{j-1} initial elements of subsections (of the pth section of $\{k_r\}$) of rank j. Clearly, each of the sequences $\{k_{fr}\}$ then determines a neighborhood E_f of X, and the family $\{E_f\}$ has the required monotonicity.

To complete the proof, let f be any fixed number in $[0, 1]$. Clearly, there exists a bounded sequence ξ which wanders slowly over $\{k_{fr}\}$ but does not wander slowly over any sequence $\{k_{gr}\}$ with $g < f$. In other words, there exists a sequence y which apes a sequence x $(x \in X)$ over $\{k_{fr}\}$, but does not ape it over $\{k_{gr}\}$ if $g < f$. Finally, the sequence $\{k_{0r}\}$ determines a neighborhood of Y which meets none of the neighborhoods E_g $(g < f)$, and the proof is complete.

THEOREM 10. *If* E *is a neighborhood of a point* X *in* $m/L - \{C\}$, *there exists a family* $\{E_f\}$ $(0 < f \leq 1)$ *of neighborhoods of* X *such that each of the sets* E_f *contains an open set which meets none of the sets* E_g $(0 < g \leq 1, g \neq f)$.

The proof is similar to that of Theorem 9, except that the family of sequences $\{k_{fr}\}$ must be constructed in such a way that each of the sequences contains a sequence of long blocks of elements such that no other sequence of the family $\{k_{fr}\}$ contains infinitely many elements from these blocks. We omit the details.

THEOREM 11. *Every open set in* $m/L - \{C\}$ *is the union of* $2^{\aleph_0}$ *disjoint open sets.*

Let E^* be an open set in $m/L - \{C\}$, and x a sequence such that $X \in E^*$. Let $\{k_r\}$ denote a fixed sequence which determines a neighborhood E of X $(E \subset E^*)$, in the sense of Theorem 2; and let $\{E_f\}$ denote the family of neighborhoods of X constructed in the proof of Theorem 9. Let the subsets G_f $(0 \leq f \leq 1)$ of E be defined by the rule that, for each point Y in E.

$$Y \in G_0 \text{ provided } Y \in E_g \text{ for all } g \ (0 < g \leq 1),$$

$$Y \in G_1 \text{ provided } Y \notin E_g \text{ if } 0 \leq g < 1,$$

$$\text{if } 0 < f < 1, \text{ then } Y \in G_f \text{ provided}$$

$$Y \in E_g \text{ if } f < g < 1 \text{ and } Y \notin E_g \text{ if } 0 < g < f.$$

Clearly, the sets G_f $(0 \leq f \leq 1)$ are mutually disjoint and cover the set E^*.

To show that each G_f is open, suppose first that $0 < f < 1$, and let Y denote any point in G_f. Then there exists a sequence $y = \{x_k \xi_k\}$ $(y \in Y,\ x \in X)$ which apes x over $\{k_{gr}\}$ if $g > f$ but not if $g < f$. By the diagonal process, we can construct a subsequence $\{h_j\}$ of $\{k_{gr}\}$ such that no divergent sequence which apes y over $\{h_j\}$ apes x over $\{k_{gr}\}$ if $g < f$. The sequence $\{h_j\}$ then determines a neighborhood H of Y which meets none of the sets E_g $(g < f)$. Also, if $z \in Z$ $(Z \in H)$, then z apes x over $\{k_{gr}\}$ whenever $g > f$, and therefore $Z \notin E_g$ when $g > f$. This concludes the proof for the case where $0 < f < 1$; the modifications needed for the two cases $f = 0$ and $f = 1$ present no special difficulties.

REFERENCES

1. R. P. Agnew, *Convergence fields of methods of summability*, Ann. of Math. (2) 46 (1945), 93-101.

2. A. L. Brudno, *Summation of bounded sequences by matrices*, Mat. Sbornik (N. S.) 16 (1945), 191-247.

3. J. Copping, *Conditions for a* K-*matrix to evaluate some bounded divergent sequences*, J. London Math. Soc. 32 (1957), 217-227.

4. V. Darevsky, *On intrinsically perfect methods of summation*, Izvestiya Akad. Nauk SSSR. Ser. Mat. 10 (1946), 97-104.

5. P. Erdös and P. C. Rosenbloom, *Toeplitz methods which sum a given sequence*, Bull. Amer. Math. Soc. 52 (1946), 463-464.

6. C. Goffman and G. M. Petersen, *Consistent limitation methods*, Proc. Amer. Math. Soc. 7 (1956), 367-369.

7. S. Mazur and W. Orlicz, *Sur les méthodes linéaires de sommation*, C. R. Acad. Sci. Paris 196 (1933), 32-34.

8. ———, *On linear methods of summability*, Studia Math. 14 (1954), 129-160.

9. G. M. Petersen, *Summability methods and bounded sequences*, J. London Math. Soc. 31 (1956), 324-326.

10. L. L. Silverman and O. Szasz, *On a class of Nörlund matrices*, Ann. of Math. (2) 45 (1944), 347-357.

11. H. Steinhaus, *Quelques remarques sur la généralisation de la notion de limite*, Prace Mat.-Fiz. 22 (1911), 121-134.

12. K. Zeller, *Allgemeine Eigenschaften von Limitierungsverfahren*, Math. Z. 53 (1950-51), 463-487.

13. ———, *Faktorfolgen bei Limitierungsverfahren*, Math. Z. 56 (1952), 134-151.

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