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ON THE PRODUCT \( \prod_{k=1}^{n} (1 - z^{a_k}) \)
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Consider the product $\prod_{k=1}^{n} (1 - z^{a_{k}})$ where $a_1 \leq a_2 \leq \cdots \leq a_n$ are positive integers. Put

$$\max_{|z|=1} \left| \prod_{k=1}^{n} (1 - z^{a_{k}}) \right| = M(a_1, a_2, \ldots, a_n), \quad f(n) = \min_{a_1, a_2, \ldots, a_n} M(a_1, a_2, \ldots, a_n).$$

Clearly $M(a_1, \ldots, a_n) \leq 2^n$ (equality if and only if $(a_1, a_2, \ldots, a_n) = 1$ or $a_1 = a_2 = \cdots = a_n = 1$). The determination of $f(n)$ seems to be a very difficult question, and even a good estimation of $f(n)$ does not seem easy. In the present note we are going to prove that $f(n) \sim n^{-1}$ as $n \to \infty$, and it seems possible that a refinement of our method would give \( \exp z = e^z \)

$$f(n) < \exp(n^{1-c})$$

for some $c < 1$. The lower bound $f(n) > \sqrt{2n}$ is nearly trivial, and we are unable at present to do any better.

We want to remark that it is easy to show that

$$\lim_{n=\infty} [M(1, 2, \ldots, n)]^{1/n}$$

exists and is between 1 and 2.

Put $z = e^{2\pi i \alpha}$, $< \alpha >= |1 - e^{2\pi i \alpha}|$. Several further questions can be asked. It is not difficult to prove that for almost all $\alpha$ (almost all means except a set of Lebesgue measure 0)

$$\lim_{n=\infty} \prod_{k=1}^{n} < k \alpha > = 0.$$
We only outline the proof of (1). A special case of a well known theorem of Khintchine states that for almost all $\alpha$ there is an infinite sequence of integers $p_n$ and $q_n$ satisfying

$$\left| \alpha - \frac{p_n}{q_n^2} \right| = o \left( \frac{1}{q_n^2 \log q_n} \right).$$

A simple computation then shows that

$$\lim \prod_{k=1}^{q_n} \langle k \alpha \rangle = 0.$$

Perhaps (1) holds for all $\alpha$.

It is easy to see that

$$\lim \prod_{k=1}^{q_n} \langle k \alpha \rangle = \infty.$$

holds for almost all $\alpha$. (Clearly (3) can not hold for all $\alpha$, e.g. it fails if $\alpha$ is rational). To see this we observe that a simple computation shows that if the $q_n$ are the integers satisfying (2) then

$$\lim \prod_{k=1}^{q_n-1} \langle k \alpha \rangle = \infty.$$

Perhaps one could determine how fast (1) tends to 0 and (3) tends to $\infty$ for almost all $\alpha$.

Is it true that for all $\alpha$

$$\lim \max_{|z|=1} \prod_{k=1}^{n} |z - e^{2\pi i k \alpha}| = \infty?$$

An old conjecture of P. Erdős which would imply (4) states as follows: Let $z_1, z_2, \ldots$ be any infinite sequence satisfying $|z_i| = 1$.

Then

$$\lim \max_{|z|=1} \prod_{l=1}^{n} |z - z_i| = \infty.$$

On the other hand a simple computation shows that for the $q_n$ satisfying (2)

$$\lim \max_{|z|=1} \prod_{k=1}^{q_n} |z - e^{2\pi i k \alpha}| = 2,$$

and perhaps

$$\lim \max_{|z|=1} \prod_{k=1}^{n} |z - e^{2\pi i k \alpha}| < \infty$$

for all irrational $\alpha$ (it certainly is $\infty$ for rational $\alpha$).
Finally we pose the following problem: Let \( a_1 < a_2 < \ldots \) be any infinite sequence of integers. Is it true that for almost all \( \alpha \)
\[
\lim_{n \to \infty} \prod_{k=1}^{n} <a_k \alpha> = \infty, \quad \lim_{n \to \infty} \prod_{k=1}^{n} <a_k \alpha> = 0?
\]

Throughout this paper \( 0 \leq \alpha < 1 \) and \( c_1, c_2, \ldots \) will denote positive absolute constants, \( |\theta| < 1 \) (and the \( \theta \)'s appearing are not necessarily the same).

**LEMMA 1.** Let
\[
\alpha = \frac{p}{q} + \frac{\theta}{q^2}, \quad (p, q) = 1.
\]

Then for every \( l \)
\[
\prod_{t=l+1}^{l+q} <t\alpha> < q^{a_l}.
\]

If \( \theta = 0 \) the product in (7) is 0, hence (7) holds. Thus we can assume \( \theta \neq 0 \). Order the numbers \( e^{2\pi i \alpha}, l+1 \leq t \leq l+q \) according to the size of their arguments and denote them by \( z_1, z_2, \ldots, z_q \) \((0 < \arg z_1 < \arg z_2 < \ldots < \arg z_q < 2\pi, i.e. (6) implies that the \( z \)'s are all different). From (6) we have
\[
\arg z_k = 2\pi \left( \frac{k}{q} + \frac{\theta}{q^2} \right) \quad (k = 1, 2, \ldots, q).
\]

Put \( y_k = e^{2\pi i (k-1/2)/q} \). From (8) we evidently have
\[
|1 - z_k| < |1 + y_k| \left( 1 + c_2 \left( \frac{1}{k} + \frac{1}{q + 1 - k} \right) \right).
\]

Now from \( \prod_{k=1}^{q} |1 - y_k| = 2, \left( \prod_{k=1}^{q} |1 - y_k| \right) \) is simply the value at \( z = 1 \) of
\[
(z^{2k} - 1)/(z^k - 1) = z^k + 1 \]
and (9) we have
\[
\prod_{k=1}^{q} |1 - z_k| < 2 \prod_{k=1}^{q} \left( 1 + c_2 \left( \frac{1}{k} + \frac{1}{q + 1 - k} \right) \right) < q^{a_1},
\]
which proves the Lemma.

**THEOREM 1.** To every \( \varepsilon \) there exists an \( n_0(\varepsilon) \) and \( A = A(\varepsilon), B = B(\varepsilon) \) so that for every \( n > n_0(\varepsilon) \) and every \( \alpha \) which does not satisfy one of the
Inequalities

\[(10) \quad \frac{1}{Bn} < \left| \frac{\alpha - p}{q} \right| \leq \frac{1}{en} \quad \text{for some } 0 \leq p < q \leq A\]

we have

\[(11) \quad \prod_{t=1}^{n} < t\alpha > < (1+\varepsilon)^n.\]

Theorem 1 means that (11) is satisfied except if \(\alpha\) can be approximated "well" but not "too well" by rational fractions with "small" denominators.

Assume first that \(\alpha\) is such that for every \(p\) and \(q \leq A\)

\[(12) \quad \left| \frac{\alpha - p}{q} \right| < \frac{1}{en}.\]

By a well known theorem of Dirichlet there exists a \(q \leq en\) for which

\[(13) \quad \left| \frac{\alpha - p}{q} \right| < \frac{1}{qen} < \frac{1}{q^2} \quad (p, q) = 1.\]

By (12) \(q > A\). Put \(uq \leq n < (u+1)q\). Then we have by \(< t\alpha > \leq 2\) and our Lemma (since \(2^q < 1 + \varepsilon\) for small \(\varepsilon\) and \(q^{1/q} < A^{1/A}\) for \(q > A > \varepsilon\))

\[(14) \quad \prod_{i=1}^{n} < t\alpha > < \frac{2^q q^e}{q} < \frac{2^qn}{q} < \frac{2^en}{A} \quad < (1+\varepsilon)^n\]

if \(A > A(\varepsilon)\) is large enough.

If for some \(q \leq A\), \(\left| \frac{\alpha - p}{q} \right| < \frac{1}{en}\) then by (10) we can assume that

\(\left| \frac{\alpha - p}{q} \right| < \frac{1}{Bn}\). But then the arguments of the numbers \(e^{2\pi i(nq + \alpha)}\), \(\nu < u\), \(1 < l < q\) \((uq \leq n < (u+1)q)\) differ from the corresponding \(q\)-th roots of unity by less than \(1/B\). Thus for \(B\) sufficiently large a simple computation gives

\[\prod_{\nu q \leq l < (\nu+1)q} < t\alpha > < \frac{1}{2},\]

or

\[(15) \quad \prod_{t=1}^{n} < t\alpha > = \prod_{t=1}^{uq} < t\alpha > \prod_{uq < t < n} < t\alpha > < \left(\frac{1}{2}\right)^u 2^q < \left(\frac{1}{2}\right)^{n/A} 2^A < 1\]

and Theorem 1, follows from (14) and (15).
Next we prove

THEOREM 2.

$$\lim f(n)^{1/n} = 1.$$ 

Let $$m^2 \leq n < (m+1)^2$$. Consider the product

$$g_n(z) = \prod_{k=1}^{m} \prod_{l=1}^{m} (1 - z^{a_{kl}}) (1 - z)^{a - m^2}.$$ 

In other words, $$a_1 = a_2 = \ldots = a_{n-m^2} = 1$$ and the other $$a$$'s are the integers $$2^k l, 1 \leq k \leq m, 1 \leq l \leq m$$. To prove Theorem 2 it will be sufficient to show that

$$(16) \quad \lim_{|z|=1} \max |g_n(z)|^{1/n} = 1.$$ 

We evidently have for $$|z| \leq 1$$

$$|1 - z|^{|n-m^2|} \leq 2^{n^2}$$ (i.e. $$n - m^2 \leq 2(\sqrt{n})^2$$).

Thus to prove (16) it will suffice to show that for every $$\varepsilon$$ if $$m > m_0(\varepsilon)$$

$$(17) \quad \max_{|z|=1} \prod_{k=1}^{m} \prod_{l=1}^{m} |1 - z^{2k\alpha}| = \max \prod_{0 \leq \alpha \leq 1} \prod_{k=1}^{m} < 2^{k} \lambda > < (1+2\varepsilon)^{m^2}.$$ 

Consider the numbers $$2^k \alpha = \alpha_k, 1 \leq k \leq m$$. We claim that only $$o(m)$$ of them satisfy (10). Since $$q \leq A$$ it will suffice to show that only $$o(m)$$ of them satisfy (10) for a fixed $$q$$.

Suppose in fact that $$\alpha_s$$ satisfies (10) for a certain $$q$$. Then we have

$$|\alpha_k - p/q| = b_k/n$$ where $$1/B \leq |b_k| \leq 1/\varepsilon$$. Also $$|\alpha_{k+1} - p'/q| = 2b_k/n$$ where $$p' = 2p$$ (mod $$q$$). Thus (10) can be satisfied for at most $$\log B/\varepsilon + 1$$ consecutive $$\log 2$$ values of $$k$$ and these are followed by at least $$c_\varepsilon \log n$$ values of $$k$$ for which (10) is not satisfied for this particular value of $$q$$ applying this argument for all the $$k \leq m$$ which satisfy (10) we obtain that (10) is satisfied for only $$o(m)$$ values of $$k$$, as stated.

Now we can prove (17). Write

$$\prod_{k=1}^{m} \prod_{l=1}^{m} < 2^k \alpha > = \prod_{l=1}^{m} \prod_{k}^{m} < 2^k \alpha > \prod_{l=1}^{m} \prod_{k}^{m} < 2^k \lambda >.$$
Thus $\prod_{k=1}^{m} 2^k / \alpha > 2^m$, thus by what we just proved

$$\prod_{k=1}^{m} \prod_{l=1}^{\alpha} 2^k / \alpha > 2^m.$$  

By Theorem 1 we have for every $k$ in $\Pi_k$

$$\prod_{l=1}^{\alpha} 2^k / \alpha > (1 + \varepsilon)^m.$$  

Thus

$$\prod_{k=1}^{m} \prod_{l=1}^{\alpha} 2^k / \alpha > (1 + \varepsilon)^m.$$  

(18) and (19) implies (17), and thus Theorem 2 is proved.

**THEOREM 3.**

$$f(n) \geq \sqrt{2n}.$$  

To prove Theorem 3 write

$$\prod_{i=1}^{n} (1 - x^{a_i}) = \sum_{i} x^{b_i} - \sum_{i} x^{c_i}, \quad b_1 < b_2 < \ldots; c_1 < c_2 < \ldots.$$  

First we show that

$$\sum_{i} b_i^p = \sum_{i} c_i^p, \quad p = 0, 1, \ldots, n - 1.$$  

To show (20) observe that 1 is an $n$-fold root of $\prod_{i=1}^{n} (1 - x^{a_i})$. Thus

$$f^{(p)}(1) = 0 \quad \text{for} \quad p = 0, 1, \ldots, n - 1; \quad \text{or}$$

$$\sum_{i} b_i (b_i - 1) \ldots (b_i - p + 1) = \sum_{i} c_i (c_i - 1) \ldots (c_i - p + 1), \quad p = 0, 1, \ldots, n - 1,$$

which implies (20). From (20) we immediately obtain that at least $n$ $b$'s and $n$ $c$'s do not vanish which implies Theorem 3 by Parseval's equality.

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