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Remarks on number theory I
On primitive α-abundant numbers

by

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Denote by $\sigma(n)$ the sum of divisors of n. It is well known that $\sigma(n)/n$ has a continuous distribution function, i. e. for every c the density of integers satisfying $\sigma(n)/n \leqslant c$ exists and is a continuous function of c whose value $\to 1$ as $c \to \infty$. This result was first proved by Davenport [1], Behrend and Chowla. Thus in particular the density of abundant numbers exists (a number is abundant if $\sigma(n)/n \geqslant 2$). I [2] have proved the existence of this density by proving that the sum of the reciprocals of the primitive abundant numbers converges (a number m is called *primitive abundant* if $\sigma(m)/m \geqslant 2$ but for every proper divisor d of m, $\sigma(d)/d < 2$). More generally we shall say that m is *primitive a-abundant* if $\sigma(m)/m \geqslant a$ but, for every proper divisor d of m, $\sigma(d)/d < a$. I observed some time ago that it is not true that the sum of the reciprocals of the primitive a-abundant numbers converges for every a. It will be clear from our proof that if a can be approximated very well by numbers of the form $\sigma(n)/n$ then the sum of the reciprocals of the primitive a-abundants will diverge.

Let $p_1, p_2, \ldots$ be an infinite sequence of primes satisfying $p_{k+1} > e^{e^{p_k^2}}$. Put

$$a = \prod_{k=1}^{\infty}\left(1+\frac{1}{p_k}\right) = \lim_{k\to\infty}\frac{\sigma(p_1p_2\cdots p_k)}{p_1p_2\cdots p_k}.$$

A simple computation shows that for every k the integers

$$p_1p_2\cdots p_kp, \quad p_k < p < p_{k+1}$$

are primitive a-abundant. From

$$\sum_{p\leqslant x}\frac{1}{p} = (1+o(1))\log\log x$$

we have

$$\sum_{p_k<p<p_{k+1}} \frac{1}{p} > \frac{1}{2} p_k^2,$$

further by the definition of the p's $p_k > p_1 p_2 \dots p_{k-1}$. Thus

$$\sum_{p_k<p<p_{k+1}} \frac{1}{p_1 p_2 \dots p_k p} > \frac{1}{2},$$

which clearly implies that the sum of the reciprocals of the primitive a-abundants diverges. A simple argument shows that the a's for which the sum of the reciprocals of the primitive a-abundants diverges form an everywhere dense G_δ in $(1, \infty)$, i. e. they are the countable intersection of dense open sets. But it is not difficult to show that they have measure 0, in fact they must be Liouville numbers (a number γ is called a *Liouville number* if γ is irrational and $|\gamma - a/b| < 1/b^n$ is solvable in integers a and b for every n). We shall not give a proof of this result.

Denote by $N_a(x)$ the number of the primitive a-abundants not exceeding x. I [3] have proved that $(\exp z = e^z)$

$$\frac{x}{\exp\left(25(\log x \log\log x)^{1/2}\right)} < N_2(x) < \frac{x}{\exp\left(\frac{1}{8}(\log x \log\log x)^{1/2}\right)}. \tag{1}$$

I can show that for every a ($c_1, c_2, \dots$ denote suitable positive constants)

$$N_a(x) > \frac{x}{\exp\left(c_1(\log x \log\log x)^{1/2}\right)}, \tag{2}$$

and that for every a and an infinite sequence $x_n \to \infty$

$$N_a(x_n) < \frac{x_n}{\exp\left(c_2(\log x \log\log x)^{1/2}\right)}. \tag{3}$$

Also if a is not a Liouville number then for a certain $c_3 = c_3(a)$

$$N_a(x) < \frac{x}{\exp\left(c_3(\log x \log\log x)^{1/2}\right)} \tag{4}$$

for all $x > 0$. I am not going to give the details of the proof of (2), (3) and (4) since the results do not seem to me to be very interesting and the proof is similar to that of (1).

I shall prove in full detail the following

THEOREM.

$$N_a(x) = o\left(\frac{x}{\log x}\right). \tag{5}$$

The proof of our Theorem will be in many ways similar to that of (1). It is easy to see that (5) is best possible in the following sense: if $g(x) \to \infty$ as slowly as we like, there always exists an a so that for infinitely many x

$$N_a(x) > \frac{x}{g(x)\log x}. \tag{6}$$

The proof of (6) can be left to the reader since it is almost identical with the proof that for a suitable a the sum of the reciprocals of the primitive a-abundants diverges.

Now we prove (5). Denote the primitive a-abundant numbers by $m_1 < m_2 < \dots$ First of all we shall show that it will suffice to consider the m_i's not exceeding x which satisfy the following properties:

I. $\dfrac{x}{(\log x)^2} < m_i$,

II. $v(m_i) < 10\log\log x$, where $v(m)$ denotes the number of distinct prime factors of m,

III. if $p^a | m_i$ and $a > 1$ then $p^a < (\log x)^{10}$,

IV. the greatest prime factor of m_i is greater than $x^{1/(\log\log x)^2}$.

To see this we shall show that the number of integers which does not satisfy any of these conditions is $o(x/\log x)$. This is trivial for I. To show it for II we observe that

$$\sum_{n=1}^{x} 2^{v(n)} < \sum_{k=1}^{x} \frac{x}{k} < 2x\log x.$$

Thus the number of integers not exceeding x which do not satisfy II is less than

$$2x\log x/2^{10\log\log x} = o(x/\log x).$$

If $p^a | n$, $a > 1$, then p^a or p^{a-1} is a square and thus n is divisible by a square greater than $(\log x)^{2a/3}$. Hence every integer which does not satisfy III is divisible by a square not less than $(\log x)^{20/3}$. Thus the number of integers which do not satisfy III is less than

$$x \sum_{k > (\log x)^{10/3}} \frac{1}{k^2} = o\left(\frac{x}{\log x}\right).$$

Let

$$n \leqslant x, \quad n = \prod_{i=1}^{x} p_i^{a_i}.$$

Assume that n does not satisfy IV. We can assume that n satisfies II and III, whence

$$n < (x^{1/(\log\log x)^2})^{10\log\log x} = o\left(\frac{x}{\log x}\right),$$

which proves that the number of integers not satisfying IV is $o(x/\log x)$.

Henceforth we shall assume that our primitive α-abundant numbers m_i satisfy the conditions I, II, III and IV. Put

$$m_i = A_i B_i \tag{7}$$

where all prime factors of A_i are $\leqslant (\log x)^{10}$ and all prime factors of B_i are $> (\log x)^{10}$. By II and III we have

$$A_i < (\log x)^{10\log\log x} \tag{8}$$

and by (8) and I we have $B_i > 1$.

Now we split the m_i into two classes. In the first class are the m_i for which B_i is not a prime. Write (by II, $(p_j^{(i)})^2 \nmid B_i$)

$$B_i = p_1^{(i)} p_2^{(i)} \dots p_{l_i}^{(i)},$$

where by IV

$$(\log x)^{10} < p_1^{(i)} < \dots < p_{l_i}^{(i)}, \quad p_{l_i}^{(i)} > x^{1/(\log\log x)^2}. \tag{9}$$

Now we split the m_i of the first class into two subclasses. In the first subclass are the m_i with

$$p_1^{(i)} < x^{1/4(\log\log x)^2}. \tag{10}$$

We shall show that if (10) is satisfied then the integers

$$\frac{m}{p_1^{(i)}} \tag{11}$$

are all different, and if this is accomplished then it will follow from (9) that the number of integers of the first subclass is less than $x/(\log x)^{10} = o(x/\log x)$.

If the integers (11) were not all different we should have

$$\frac{m_i}{p_1^{(i)}} = \frac{m_j}{p_1^{(j)}}, \quad p_1^{(i)} \neq p_1^{(j)} \quad \text{(assume say } p_1^{(j)} < p_1^{(i)}\text{)}.$$

Thus

$$\sigma\left(\frac{m_i}{p_1^{(i)}}\right)\frac{p_1^{(i)}}{m_i} = \sigma\left(\frac{m_j}{p_1^{(j)}}\right)\frac{p_1^{(j)}}{m_j} \quad \text{or} \quad \frac{\sigma(m_i)}{m_i}\cdot\frac{p_1^{(i)}}{p_1^{(i)}+1} = \frac{\sigma(m_j)}{m_j}\cdot\frac{p_1^{(j)}}{p_1^{(j)}+1},$$

whence

$$(12)\qquad \gamma_{i,j} = \frac{p_1^{(i)}(p_1^{(j)}+1)}{p_1^{(j)}(p_1^{(i)}+1)} = \frac{\sigma(m_i)}{m_i}\left(\frac{\sigma(m_j)}{m_j}\right)^{-1}.$$

Now since the m's are primitive α-abundant we have

$$\frac{\sigma(m_i)}{m_i} \geqslant \alpha, \quad \sigma\left(\frac{m_i}{p_{l_i}^{(i)}}\right)\frac{p_i}{m_i} = \frac{\sigma(m_i)}{m_i}\left(1+\frac{1}{p_{l_i}^{(i)}}\right)^{-1} < \alpha$$

or by (9)

$$(13)\qquad \alpha \leqslant \frac{\sigma(m_i)}{m_i} < \alpha(1+x^{-1/(\log\log x)^2}),$$

and the same holds with m_j replacing m_i. Further from (9), (10) and (12)

$$(14)\qquad \gamma_{i,j} \geqslant 1+\frac{1}{p_1^{(j)}(p_1^{(i)}+1)} > 1+\frac{1}{2}x^{-1/2(\log\log x)^2}.$$

On the other hand from (12) and (13)

$$(15)\qquad \gamma_{i,j} < 1+x^{-1/(\log\log x)^2}.$$

(15) clearly contradicts (14); this shows that the integers (11) are all different and this disposes of the first subclass.

Now we deal with the numbers of the second subclass. For these numbers we have

(16) $\quad m_i = A_iB_i$, B_i not a prime, all prime factors of B_i are $> x^{1/4(\log\log x)^2}$.

I now show that for all m_i of the second subclass

$$(17)\qquad \sigma(A_i)/A_i = C < \alpha.$$

Assume that (17) does not hold. $\sigma(A_i)/A_i < \alpha$ is clear since m is primitive α-abundant. Assume thus that for some m_1 and m_2 of the second subclass we have $m_1 = A_1B_1$, $m_2 = A_2B_2$, $\sigma(A_1)/A_1 < \sigma(A_2)/A_2$. But then by (8)

$$(18)\qquad \frac{\sigma(A_2)}{A_2} - \frac{\sigma(A_1)}{A_1} \geqslant \frac{1}{A_1A_2} > (\log x)^{-20\log\log x}$$

$$\text{or}\quad \frac{\sigma(A_1)}{A_1} < \alpha-(\log x)^{-20\log\log x}.$$

Now by (16) (the number of prime factors of B_i is less than $4(\log\log x)^2$)

$$(19)\qquad \frac{\sigma(B_1)}{B_1} < (1+x^{1/4(\log\log x)^2})^{4(\log\log x)^2} < 1+(\log x)^{-30\log\log x}.$$

But by (18) and (19)

$$a \leqslant \frac{\sigma(m_1)}{m_1} = \frac{\sigma(A_1)}{A_1} \cdot \frac{\sigma(B_1)}{B_1} < a,$$

an evident contradiction thus; (17) is proved.

Now let p_1 be the smallest prime factor of all the B's which belong to the m's of the second subclass. We have to split these m's again into two classes (sub-subclasses). In the first class are those m's for which the least prime factor $p_1^{(i)}$ of B_i satisfies

$$x^{1/4(\log\log x)^2} < p_1 \leqslant p_1^{(i)} < p_1\left(1+\frac{1}{\log x}\right). \tag{20}$$

The number of the m's of the first class is clearly (by (20) and the prime number theorem, or a more elementary inequality) less than

$$x \sum_{p_1 \leqslant p < p_1(1+1/\log x)} \frac{1}{p} < \frac{cx(\log\log x)^2}{(\log x)^2} = o\left(\frac{x}{\log x}\right). \tag{21}$$

For the m_i of the second class we have

$$p_1^{(i)} \geqslant p_1\left(1+\frac{1}{\log x}\right). \tag{22}$$

First we show that for every B_i

$$\frac{\sigma(B_i)}{B_i} > 1 + \frac{1}{p_1}. \tag{23}$$

To see this we only have to remark that p_1 is a prime factor of some $m_j = A_j B_j$. Thus since B_j is not a prime

$$\frac{\sigma(A_j p_1)}{A_j p_1} = \left(1+\frac{1}{p_1}\right)\frac{\sigma(A_j)}{A_j} < a \leqslant \frac{\sigma(A_j)}{A_j} \cdot \frac{\sigma(B_i)}{B_i},$$

which proves (23). From (23) and (16) we have

$$\left(1+\frac{1}{p_1^{(i)}}\right)^{4(\log\log x)^2} > \frac{\sigma(B_i)}{B_i} > 1 + \frac{1}{p_1}.$$

Thus

$$p_1\left(1+\frac{1}{\log x}\right) \leqslant p_1^{(i)} < 10(\log\log x)^2 p_1. \tag{24}$$

Next we estimate $p_2^{(i)}$ ($p_1^{(i)} < p_2^{(i)} < \ldots$ are the prime factors of B_i). We have by (23) and (24)

$$\left(1+\frac{1}{p_1}\right) < \frac{\sigma(B_i)}{B_i} < \left(1+\frac{1}{p_1^{(i)}}\right)\left(1+\frac{1}{p_2^{(i)}}\right)^{4(\log\log x)^2}.$$

Thus by (24)

$$\left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_1(1+(\log x)^{-1})}\right)^{-1} \leqslant \left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_1^{(i)}}\right)^{-1} < \left(1+\frac{1}{p_2^{(i)}}\right)^{4(\log\log x)^2},$$

or by a simple computation (for sufficiently large X)

$$p_2^{(i)} < 10\log x(\log\log x)^2 p_1 < (\log x)^2 p_1.$$

Thus B_i has at least two prime factors in the interval $(p_1, (\log x)^2 p_1)$. Hence the number of m_i's of the second class is less than (q, r, s are primes)

$$x \sum_{p_1<q<r<p_1(\log x)^2} \frac{1}{qr} < \Bigg(\sum_{p_1<s<p_1(\log x)^2} \frac{1}{s}\Bigg)^2 < c_5 x\left(\frac{\log\log x}{\log p_1}\right)^2 \tag{25}$$

$$< c_6 x\frac{(\log\log x)^4}{(\log x)^2} = o\left(\frac{x}{\log x}\right).$$

(21) and (25) shows that the number of integers of the second subclass is also $o(x/\log x)$. In fact, the number of primitive a-abundants we have considered so far is easily seen to be $o(x/(\log x)^{2-\varepsilon})$.

Finally we consider the m's of the second class. Here

$$m_i = A_i p_i. \tag{26}$$

From (8) and I it follows that it suffices to consider the m_i satisfying $p_i > x^{1/2}$, but then we can again assume that (17) holds, i. e. that $\sigma(A_i)/A_i = C < a$ (the proof is the same as previously). But then the number of integers of the second class equals (by (8))

$$\sum{}' \pi\left(\frac{x}{A_i}\right) + o\left(\frac{x}{\log x}\right) = \frac{x}{\log x}\sum{}' \frac{1}{A_i} + o\left(\frac{x}{\log x}\right)$$

where the dash indicates that the summation is extended over the A_i satisfying $\sigma(A_i)/A_i = C$. Thus to prove our Theorem it will suffice to show that

$$\sum{}' \frac{1}{A_i} = o(1). \tag{27}$$

Now denote by $n_1 < n_2 < \dots$ the sequence of all integers which satisfy $\sigma(n_i)/n_i = C$. Clearly

(28) $$\sum{}' \frac{1}{A_i} \leqslant \sum \frac{1}{n_i}.$$

Put $\sigma(n_1)/n_1 = a/b = C$. First we show that as $x \to \infty$, $b \to \infty$ (C and therefore the n's depend on x). If for infinitely many values of x b assumed the same value, then since there are only a finite number of choices of a $(a/b < a)$, we should have from (24)

(29) $$\frac{\sigma(n_1)}{n_1} = \frac{a}{b} < a, \quad \frac{a}{b}\left(1+\frac{1}{p_1}\right) \geqslant a, \quad p_1 > (\log x)^{10},$$

(if no such p_1 existed there would be no integers of the form (26), i. e. there would be no integers of the second class and the proof of our Theorem would be complete). But since $p_1 \to \infty$, (29) is clearly impossible, thus, $b \to \infty$ as $x \to \infty$ as stated.

Thus to complete our proof it will suffice to show that

(30) $$\sum \frac{1}{n_i} < \frac{c_7}{b^{1/2}}.$$

We write

(31) $$\sum \frac{1}{n_i} = \sum{}' + \sum{}''$$

where in $\sum'$ all the prime factors of the n_i are not greater than b and in $\sum''$ are the other n_i. Clearly for all i, $n_i \equiv 0 \pmod{b}$; thus

(32) $$\sum{}' \leqslant \frac{1}{b} \sum{}_1 \frac{1}{t} = \frac{1}{b} \prod_{p \leqslant b} \left(1+\frac{1}{p-1}\right) < c_8 \frac{\log b}{b}$$

where in $\sum_1$ all prime factors of t are not exceeding b. Now let n_i be in $\sum''$ and let p_i be the greatest prime factor of n_i. Clearly $p_i > b$. But since $\sigma(n_i)/n_i = a/b$, we must have $\sigma(n_i) \equiv 0 \pmod{p_i}$. Therefore

$$n_i \equiv 0 \pmod{q_i^a}, \quad \sigma(q_i^a) \equiv 0 \pmod{p_i}, \quad a > 1$$

or

$$q_i^a > \tfrac{1}{2} p_i.$$

($a > 1$ follows from the fact that p_i was the greatest prime factor of n_i).

Thus

$$\sum_2 < \sum_{p_i > b} \frac{1}{p_i} \sum_2 \frac{1}{q^a} \sum_3 \frac{1}{t}$$

where in $\sum_2$, $q^a > \frac{1}{2}p_i$ and $a > 1$ and in $\sum_3$ all prime factors of t are not greater than p . Thus finally

$$(33) \quad \sum_2 < \sum_{p_i > b} \frac{1}{p_i} \sum_2 \frac{1}{q^a} \prod_{p \leqslant p_i} \left(1 + \frac{1}{p-1}\right) < \sum \frac{c_8 \log p_i}{p_i} \sum_2 \frac{1}{q^a}$$

$$< c_9 \sum_{p_i > b} \frac{\log p_i}{p_i^{3/2}} < \frac{c_{10}}{b^{1/2}}.$$

(31), (32) and (33) prove (30) and hence the proof of our Theorem is complete.

References

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