A CONSTRUCTION OF GRAPHS WITHOUT TRIANGLES HAVING PRE-ASSIGNED ORDER AND CHROMATIC NUMBER

P. Erdős and R. Rado*

1. Introduction and statement of result.

The chromatic number $\chi(\Gamma)$ of a combinatorial graph $\Gamma$ is the least cardinal number $\mu$ such that the set of nodes of $\Gamma$ can be divided into $\mu$ subsets so that every edge of $\Gamma$ joins nodes belonging to different subsets. It is known† that corresponding to every finite $\mu$ there exists a finite graph $\Gamma_\mu$ without triangles satisfying $\chi(\Gamma_\mu) = \mu$. In [1], Theorem 2, we have extended this result to transfinite values of $\mu$. For every graph $\Gamma$ the order $\phi(\Gamma)$, i.e. the cardinal of the set of nodes of $\Gamma$, satisfies $\phi(\Gamma) \geq \chi(\Gamma)$. The construction used in [1] was of considerable complexity and did not allow us to prove that it was most economical, i.e. that it leads to a graph $\Gamma_\mu$ such that $\phi(\Gamma_\mu) = \mu$. This equation was only established ([1], Theorem 3) when essential use was made of a form of the general continuum hypothesis.

In the present note we describe a much simpler construction of such a graph $\Gamma_\mu$ and we shall at the same time prove, without using the continuum hypothesis, that our new graph $\Gamma_\mu$ satisfies $\phi(\Gamma_\mu) = \chi(\Gamma_\mu) = \mu$. Trivially, for instance by adding isolated nodes to the graph, we can make its order equal to any given cardinal $\nu$ such that $\nu \geq \mu$, without changing the chromatic number or introducing any triangles.

**THEOREM.** Given $\mu \geq \aleph_0$, there is a graph $\Gamma_\mu$ without triangles such that $\phi(\Gamma_\mu) = \chi(\Gamma_\mu) = \mu$.

The proof depends on some lemmas, each a special case of a more general proposition. An essential part is played by Lemma 4, which is an adaptation of a result due to Specker [2].

2. Notation.

We use the notation set out in [1], §2. Every small letter, unless the contrary is stated, denotes an ordinal. The order type of an ordered set $A$ is denoted by $\text{tp} A$. If $A, B, \ldots$ are elements of an ordered set then the symbol $\{A, B, \ldots\} <$ denotes the set $\{A, B, \ldots\}$ and at the same time expresses the fact that $A < B < \ldots$. For a cardinal $\mu$, the partition relation†

$$\mu \rightarrow (\beta_0, \beta_1, \ldots, \beta_\mu)^{\mu} \tag{1}$$

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† [3], [4], [5].
‡ The obliteration operator * removes from a well-ordered sequence the term above which it is placed.

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expresses the fact that whenever $\text{tp} \ A = \alpha$; \[A\] = $\Sigma(v < n) K$, there is a subset $B$ of $A$ and an ordinal $v < n$ such that $\text{tp} \ B = \beta_v$; \[B\] = $K_v$. If $\theta_0 = \ldots = \vartheta_n = \beta$ we write (1) also in the form

$$\alpha \rightarrow (\beta)_{n!}.$$ 

The logical negation of (1) is denoted by

$$\alpha \rightarrow (\beta_0, \ldots, \vartheta_n)^v.$$ 

3. Lemmas.

Throughout Lemmas 1-5 we denote by $\alpha$ a fixed ordinal such that either $\alpha = \omega_0$ or $\alpha$ is of the form $\omega_{1+}$.

LEMMA 1. Let $\beta$ be an ordinal and $c$ a cardinal such that

$$\alpha \rightarrow (\beta,\varphi)^c,$$

Then $\alpha \rightarrow (\beta,\varphi)^c$.

Proof. Let $S = \{(y, x) : x < \alpha ; y < \beta \}$, and order $S$ lexicographically. Then $\text{tp} S = \alpha$. Let $\Pi = S; \Sigma = S(v \in N) S_v$. Choose any $y < \beta$. Put $A_v(y) = \{x : (x, y) \in S_v \} (v \in N)$. Then, since every $x < \alpha$ is a member of some $A_v(y), \Sigma v < \omega_{1+} \ A_v(y), \text{and by } \alpha \rightarrow (\beta,\varphi)^c \ | \ V \ | \ \text{there is an element } v(y) \text{ of } N \text{ with } \text{tp} A_v(y) \geq \alpha$. Put $B_v = \{v : v(y) = v \} (v \in N)$. Then, since $y$ can take any value less than $\beta$, $\Sigma v < \omega_{1+} \ B_v \geq \beta$. Then $\text{tp} A_v(y) \geq \alpha (y \in B_v)$, and the set $D = \{(y, x) : x \in A_v(y) \}$ satisfies

$$D \subseteq S_v; \text{ tp } S_v \geq \text{tp } D = \alpha \beta.$$ 

This proves Lemma 1.

LEMMA 2. $\alpha \rightarrow (\beta,\varphi)^c$ for every cardinal $p$ such that $p < | \alpha |$.

Proof. We need only consider the case $\alpha = \omega_{1+}$; $p = \aleph_\lambda$. Let $[0, \alpha) = \Sigma v < \omega_{1+} \ S_v$. If for all $v < \omega_{1+}$ we have $| S_v | \leq \aleph_{\lambda+}$ then the contradiction $\aleph_{\lambda+} \leq \Sigma v < \omega_{1+} | S_v | \leq \aleph_{\lambda+}^2 \leq \aleph_\lambda$ follows. Hence there is $v_0 < \omega_{1+}$ with $| S_{v_0} | = \aleph_{\lambda+}$, and so $\text{tp} S_{v_0} = \alpha$. This proves $\alpha \rightarrow (\beta,\varphi)^c$, and Lemma 2 follows by two applications of Lemma 1.

LEMMA 3. Let $k < \omega_0$, and let $V$ be a set of vectors $(x_0, \ldots, x_k)$ with $x_0, \ldots, x_k < \alpha$, ordered lexicographically. Let $\text{tp} V = \alpha$. Then there are sets $T_v(x_0, \ldots, x_k) \subseteq [0, \alpha)$ with $\text{tp} T_v(x_0, \ldots, x_k) = \alpha (v < k; x_0, \ldots, x_k < \alpha)$ such that the relations $x \in T_v(x_0, \ldots, x_k) (v < k)$ imply $(x_0, \ldots, x_k) \in V$.

Proof. Let $x = \omega_{1+}$. The assertion holds for $k = 0$. Let $k > 1$, and use induction with respect to $k$. Put

$$f(x_0) = \{(x_1, \ldots, x_k) : (x_0, x_1, \ldots, x_k) \in V \} (x_0 < \alpha).$$
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Then $\text{tp } f(x) \leq x^{k-1} (x < \alpha)$; $\text{tp } V = \Sigma (x < \alpha) \text{tp } f(x)$.

Put

$$T_0 = \{ x : \text{tp } f(x) = \alpha^{k-1} \}.$$  

Assume that

$$\text{tp } T_0 < \alpha.$$  

Then $\text{tp } T_0 < \omega_{\lambda+1}$; $|T_0| \leq \aleph_\lambda$, and $T_0$ is not cofinal in $[0, \alpha)$. There is $\beta < \alpha$ with $T_0 \subseteq [0, \beta)$. If $k = 1$ then the contradiction

$$\alpha = \text{tp } V = \Sigma (x < \beta) \text{tp } f(x) \leq \beta$$  

follows. Now let $k \geq 2$. Then $\text{tp } f(x) \leq \alpha^{k-2} \delta(x)$ where

$$\delta(x) < \alpha; \quad |\delta(x)| \leq \aleph_\lambda \quad (\beta < x < \alpha).$$  

If $\beta \leq \gamma < \alpha$ then

$$|\delta(\beta) + \ldots + \delta(\gamma)| \leq \aleph_\lambda |\gamma| \leq \aleph_\lambda; \quad \delta(\beta) + \ldots + \delta(\gamma) < \omega_{\lambda+1} = \alpha.$$  

Hence $\sigma = \delta(\beta) + \ldots + \delta(\alpha) \leq \alpha$, and we obtain the contradiction

$$\text{tp } V < \Sigma (x < \beta) \alpha^{k-1} + \Sigma (\beta \leq x < \alpha) \alpha^{k-2} \delta(x) = \alpha^{k-1} \beta + \alpha^{k-2} \sigma$$  

$$\leq \alpha^{k-1} (\beta + 1) < \alpha^k.$$  

Hence the assumption is false, and $\text{tp } T_0 = \alpha$.

Let $x_0 \in T_0$. By induction hypothesis, applied to $f(x_0)$, there are sets

$$T_\nu(x_0, \ldots, \hat{x}_\nu) \subseteq [0, \alpha) \quad (1 \leq \nu < k; \ x_1, \ldots, \hat{x}_\nu < \alpha)$$  

with

$$\text{tp } T_\nu(x_0, \ldots, \hat{x}_\nu) = \alpha \quad (1 \leq \nu < k; \ x_1, \ldots, \hat{x}_\nu < \alpha)$$  

such that whenever

$$x_v \in T_\nu(x_0, \ldots, \hat{x}_\nu) \quad (1 \leq \nu < k)$$  

then $(x_1, \ldots, \hat{x}_\nu) \in f(x_0)$. Put

$$T_\nu(x_0, \ldots, \hat{x}_\nu) = [0, \alpha) \quad (1 \leq \nu < k; \ x_0 \in [0, \alpha) \setminus T_0; \ x_1, \ldots, \hat{x}_\nu < \alpha).$$  

Then the sets $T_\nu (\nu < k)$ satisfy the assertion of Lemma 3.

**Lemma 4.** $\alpha^3 \rightarrow (3, \alpha^3)$.  

**Proof.** Put $S = \{(x, y, z) : x, y, z < \alpha \}$ and order $S$ lexicographically. Then $\text{tp } S = \alpha^3$; $[S]^2 = K_0 + K_1$; $K_0 K_1 = \emptyset$,

$$K_0 = \left\{ \{(a_0, a_1, a_2), (b_0, b_1, b_2), (c_0, c_1, c_2)\} : a_1 < b_0 < a_2 < b_1 < \alpha \right\}.$$  

If ordinals $a_0, b_0, c_0$ satisfy

$$[\{(a_0, a_1, a_2), (b_0, b_1, b_2), (c_0, c_1, c_2)\}]^2 = K_0$$  

then the contradiction $a_2 < b_1 < c_0 < a_2$ follows.

If, on the other hand, a subset $V$ of $S$ satisfies $\text{tp } V = \alpha^3$; $[V]^2 = K_1$ then there are sets $T_\nu$ which have, for $k = 3$, the properties mentioned in Lemma 3. Then there are ordinals $a_0, b_0, c_0$ such that

$$a_0 \in T_0; \quad a_1 \in T_1(a_0) \setminus [0, a_0 + 1); \quad b_0 \in T_0 \setminus [0, a_1 + 1),$$

$$a_2 \in T_2(a_0, a_1) \setminus [0, b_0 + 1); \quad b_1 \in T_1(b_0) \setminus [0, a_2 + 1); \quad b_2 \in T_2(b_0, b_1).$$  


But then the contradiction \( \{a_0, a_1, a_2, \( (b_0, b_1, b_2) \} \in K_0(V)^2 = \emptyset \) follows. This proves Lemma 4.

**Lemma 5.** There is a graph \( \Gamma \) without triangles such that, if \( \chi(\Gamma) = e \),

\[
\phi(\Gamma) = |x|; \quad x^3 \rightarrow (x^3)_e^1.
\]

**Proof.** Let \( \alpha = \omega_{\lambda+1} \); \( tp S = \alpha^3 \). By Lemma 4 there is a partition \( \mathcal{S}^2 = K_0 + K_1 \) such that (i) there is no \( A \subset S \) such that \( tp A = 3; \mathcal{A}^2 \subset K_0 \), (ii) there is no \( B \subset S \) such that \( tp B = \alpha^3; \mathcal{B}^2 \subset K_1 \). Put \( \Gamma = (S, K_0) \). Then \( \Gamma \) has no triangle, and \( \phi(\Gamma) = |S| = |x^3| = \mathcal{S}_{\lambda+1} \). Let \( |N| = \chi(\Gamma) \). Then there is a function \( g \) from \( S \) into \( N \) such that \( g(x) = g(y) \) implies \( \{x, y\} \notin K_0 \). Then \( S = \sum (\nu \in N) S_\nu \), where \( S_\nu = \{x: g(x) = \nu \} (\nu \in N) \). Let \( \nu \in N \). If \( x, y \in S_\nu \), then \( g(x) = \nu = g(y) \); \( \{x, y\} \notin K_0 \). Hence \( |S_\nu|^2 \subset K_1 \); whence by (ii) above \( tp S_\nu < \alpha^3 \). This proves \( \alpha^3 \rightarrow (x^3)_{\nu+1} \) and completes the proof of Lemma 5.

**Proof of the Theorem.**

**Case 1.** \( a = \mathcal{S}_0 \). By Lemma 5, with \( \alpha = \omega_{\lambda+1} \), there is a graph \( \Gamma \) without triangles such that \( \phi(\Gamma) = \mathcal{S}_0 \); \( \omega_0^3 \rightarrow (\omega_0^3)_e^1 \), where \( e = \chi(\Gamma) \). By Lemma 2 it follows that \( e \geq \mathcal{S}_0 \). Hence \( \mathcal{S}_0 \leq \chi(\Gamma) \leq \phi(\Gamma) = \mathcal{S}_0 \), and we may put \( \Gamma_\alpha = \Gamma \).

**Case 2.** \( a > \mathcal{S}_0 \). Put \( M = \{b^+: \mathcal{S}_1 \leq b^+ \leq a\} \), where \( b^+ \) denotes the next larger cardinal to the cardinal \( b \). Then \( \mathcal{S}_1 \in M \); \( |M| \leq a \). Let \( c = b^+ \in M \). Then \( b = \mathcal{S}_\lambda \) for some \( \lambda \). Put \( \alpha = \omega_{\lambda+1} \). By Lemma 5 there is a graph \( \Gamma'_c \) without triangles such that \( \phi(\Gamma'_c) = \mathcal{S}_{\lambda+1}; \omega^3 \rightarrow (\omega^3)_e^1 \), where \( e = \chi(\Gamma'_c) \). Then, by Lemma 2, \( e \geq c \). We can arrange that \( \Gamma'_c = (A_c, B_c) \), where \( A_c \cap (\mathcal{S}_c \cap M) = \emptyset \). Put

\[
\Gamma_a = (\Sigma (e \in M) A_c, \Sigma (e \in M) B_c).
\]

Then \( \chi(\Gamma_a) \geq \chi(\Gamma'_c) \geq \mathcal{S}_1 \). If \( \chi(\Gamma_a) = d < a \), then \( \mathcal{S}_2 \leq d^+ \leq a \); \( d^+ \in M \), and we obtain the contradiction \( \chi(\Gamma_{d^+}) \geq \chi(\Gamma_{d+}) \geq d^+ \). Hence

\[
a \leq \chi(\Gamma_a) \leq \phi(\Gamma_a) = |\Sigma (e \in M) A_c| \leq |\Sigma (e \in M) A_c| = a(|M| \leq a),
\]

and the theorem is proved.

**References.**


The University, Birmingham.
The University, Reading.

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