ON A PROPERTY OF FAMILIES OF SETS

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1. Introduction. In this paper we are going to generalize a problem solved by MILLER in his paper [1] and prove several results concerning this new problem and some related questions. We mention here that some of our theorems (Theorems 8 and 10) have the interesting consequence that the topological product of $\aleph_k$ 1-compact spaces (Lindelöf spaces) is not necessarily k-compact for any finite k.[1]

DEF. (1.1) Let $\mathfrak{F}$ be a family of sets. $\mathfrak{F}$ is said by MILLER to possess property **B** if there exists a set B such that

$$F \cap B \neq 0 \quad \text{for every} \quad F \in \mathfrak{F},$$
$$F \subsetneqq B \quad \text{for every} \quad F \in \mathfrak{F}.$$

DEF. (1.2) Let $\mathfrak{F}$ be a family of sets. Let $p^*(\mathfrak{F})$ denote the least cardinal number p for which $\overline{\overline{F}} \leqq p$ for every $F \in \mathfrak{F}$. If $\overline{\overline{F}} \geqq p$ for every $F \in \mathfrak{F}$, we write $|\mathfrak{F}| \geqq p$. In what follows $p(\mathfrak{F}) = p$ denotes briefly that the family $\mathfrak{F}$ possesses the property

$$p^*(\mathfrak{F}) = p, \quad |\mathfrak{F}| \geqq p.$$

DEF. (1.3) Let $\mathfrak{F}$ be a family of sets and let $q \geqq 2$, $r \geqq 1$ be cardinal numbers. The family $\mathfrak{F}$ is said to possess property $\mathbf{C}(q, r)$ if $\overline{\overline{\bigcap_{F \in \mathfrak{F}'} F}} < r$ for every subfamily $\mathfrak{F}'$ of $\mathfrak{F}$, provided $\overline{\overline{\mathfrak{F}'}} \geqq q$.

NOTE. If for a family $\mathfrak{F}$ $|\mathfrak{F}| \geqq r$ and $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$, then $\mathfrak{F}$ consists of almost disjoint sets.

The result of MILLER which is our starting point can be stated as follows:

(1.4) Let p be an infinite cardinal number, n an integer ($n > 0$) and let $\mathfrak{F}$ be a family which possesses property $\mathbf{C}(p^+, n)$ such that $|\mathfrak{F}| \geqq p$. Then the family $\mathfrak{F}$ possesses property **B**.[2]

[1] In our example the spaces will be discrete ones. The generalized continuum hypothesis is used in the proof. As far as we know this result is new already for $k = 2$. This theorem should be compared with a theorem of J. Łos [3] (see Section 7).

[2] See [1], p. 35, Corollary.

To show that this result is best-possible MILLER proves the following:

(1.5) There exists a family $\mathfrak{F}$ $(p(\mathfrak{F})=\aleph_0, \overline{\overline{\mathfrak{F}}}=2^{\aleph_0})$ which possesses property $\mathbf{C}(2,\aleph_0)$ and fails to possess property $\mathbf{B}$.[3]

However, one can ask what happens if $\mathfrak{F}$ possesses property $\mathbf{C}(2,\aleph_0)$ and $|\mathfrak{F}|$ is supposed to be greater than $\aleph_0$.

On the other hand, one can sharpen property $\mathbf{B}$ as follows:

DEF. (1.6) Let $\mathfrak{F}$ be a family, s a cardinal number, $s \geqq 2$. $\mathfrak{F}$ is said to possess property $\mathbf{B}(s)$ if there exists a set B such that $F \cap B \neq 0$ and $\overline{\overline{F \cap B}} < s$ for every $F \in \mathfrak{F}$.

Our problems will be of the following kind. Let $\mathfrak{F}$ be a family of sets, and let m, p, q, r, s be cardinal numbers such that $\overline{\overline{\mathfrak{F}}} = m$, $p(\mathfrak{F}) = p$ and suppose that $\mathfrak{F}$ possesses property $\mathbf{C}(q, r)$. Under what conditions for the cardinals m, p, q, r, s has $\mathfrak{F}$ to possess the properties $\mathbf{B}$ and $\mathbf{B}(s)$, respectively?

As the easy example (3.3) will show, nothing can be said about property $\mathbf{B}(s)$ if $q>2$. The case $q=2$ contains the essential difficulty in the researches concerning the property $\mathbf{B}$ too.

The problem just stated is clearly a generalization of the problem treated in (1.4) which is a corollary of [1], Theorem 1. We remark that it would be possible to generalize in a quite similar way the theorem itself not only its corollary, however, such a generalization does not seem to need new ideas and its formulation would be very complicated.

We restricted the formulation of the general problem with the assumption $p(\mathfrak{F}) = p$ instead of MILLER's original assumption $|\mathfrak{F}| \geqq p$. This has no importance in the problems concerning property $\mathbf{B}$, however, in the problems for property $\mathbf{B}(s)$ it seems to be an essential restriction (see the remark at the end of Section 4).

2. Definitions. Notations. We use the usual notations of the set theory. We are going to list only those where there is a danger of misunderstanding.

In what follows $\mathfrak{F}, \mathfrak{G}, \ldots$ will denote families (sets of sets); capital letter will denote sets; $x, y, \ldots$ are the elements of the sets; m, t, p, q, r, s denote cardinals; $i, j, k, l, n, \ldots$ denote non-negative integers; $\alpha, \beta, \ldots$ denote ordinal numbers. Union and intersection of sets will be denoted by $\cup$ and $\cap$, respectively.

t^+ denotes the least cardinal greater than t (if t is finite, $t^+ = t+1$). t^- is the immediate predecessor of the cardinal t if it exists, if not, then $t^- = t$. (If t is finite, $t^- = t-1$ for $t>0$ and $t^- = 0$ for $t=0$.)

$\mathfrak{S}(S)$ denotes the set of all subsets of S.

[3] See [1], Theorem 3.

If $(x_\nu)_{\nu<\varphi}$ is an arbitrary sequence of type φ of not necessarily different elements x_ν, then $\{x_\nu\}_{\nu<\varphi}$ denotes the set of all x_ν's forthcoming in the sequence. This distinction will be sometimes omitted if there is no danger of misunderstanding.

Let $\varphi(x)$ be an arbitrary property of the elements of a set H. The set of all $x \in H$ which satisfy $\varphi(x)$ will be denoted by $\{x:\varphi(x)\}$. (We are going to use the logical signs $\wedge$ (and), $\vee$ (or) in the formulation of these formulas.)

The sets $\{X:X \subseteq S \wedge \overline{\overline{X}}=t\}$, $\{X:X\subseteq S \wedge \overline{\overline{X}}<t\}$ will be denoted by $[S]^t$, $[S]^{<t}$, respectively.

For an arbitrary family $\mathfrak{F}$ the set $\bigcup_{F\in\mathfrak{F}} F$ is denoted by $(\mathfrak{F})$. Other special notations concerning families will be introduced later.

For the study of the problem stated in the introduction we introduce the following symbols:

Def. (2.1) $\mathbf{M}(m,p,q,r)\to\mathbf{B}$ indicates the statement that every family $\mathfrak{F}$ which possesses property $\mathbf{C}(q,r)$ possesses property $\mathbf{B}$, provided $p(\mathfrak{F})=p$ and $\overline{\overline{\mathfrak{F}}}=m$.

Def. (2.2) $\mathbf{M}(m,p,r)\to\mathbf{B}(s)$ indicates the statement that every family $\mathfrak{F}$ which possesses property $\mathbf{C}(2,r)$ possesses property $\mathbf{B}(s)$, provided $p(\mathfrak{F})=p$ and $\overline{\overline{\mathfrak{F}}}=m$. $\mathbf{M}(m,p,q,r)\not\to\mathbf{B}$ and $\mathbf{M}(m,p,r)\not\to\mathbf{B}(s)$ denote the negations of the corresponding statements, respectively.

To exclude trivial exceptions here we assume once for all $m>0$, $p>0$, $q>1, r>0, s>1$.

We call briefly the symbols now introduced symbol-I and symbol-II, respectively.

The proof of some of our theorems makes use of the generalized continuum hypothesis or of the so-called measure hypothesis stated in [5]. These hypotheses will be cited as hypotheses $(*)$, $(**)$ and the corresponding theorems will be denoted by the same signs, respectively.

3. Preliminaries. A short summary of the content of the following sections. We briefly say that one of the symbols is monotone increasing (decreasing) in one of its variables, e. g. in m, if the fact that it is true for $m, p, q, r, (s)$ implies that it is true for $m', p, q, r, (s)$ for $m' \geqq m$ $(m' \leqq m)$, respectively. The following monotonicity properties are immediate consequences of the definitions (2.1) and (2.2):

(3.1) Both symbols are decreasing in m and r. Symbol-I is decreasing in q. Symbol-I is increasing in p, symbol-II is increasing in s.

We call attention that symbol-II is not increasing in p (see the end of Section 4).

It is also obvious that

(3. 2) $\mathbf{M}(m,p,r)\to\mathbf{B}(s)$ implies $\mathbf{M}(m,p,2,r)\to\mathbf{B}$ if $s\leqq p$. (For $s>p$ symbol-II is trivially true.)

Now we prove:

(3. 3) *Let $p\geqq\aleph_0$, $s\leqq p$ be cardinal numbers. There exists a family $\mathfrak{F}$ such that $\overline{\overline{F}}=p$, $p(\mathfrak{F})=p$, $\mathfrak{F}$ possesses property $\mathbf{C}(3,1)$ and it does not possess property $\mathbf{B}(s)$.*

PROOF. Let S be a set of power p. Let $\mathfrak{F}'$ be a system of subsets of S such that $\overline{\overline{\mathfrak{F}'}}=p$, $F_1\cap F_2=0$ for $F_1,F_2\in\mathfrak{F}'$, $F_1\neq F_2$ and $\overline{\overline{F}}=p$ for every $F\in\mathfrak{F}'$. Put $\mathfrak{F}=\{S\}\cup\mathfrak{F}'$. It is obvious that $\mathfrak{F}$ satisfies the requirements of (3. 3) for $s=p$.

(3. 3) shows that in the investigations concerning property $\mathbf{B}(s)$ the assumption $q\leqq 2$, i. e. $q=2$ is essential.

MILLER's result (1. 4) can be stated as follows:

THEOREM 1. *Suppose $p\geqq\aleph_0$, $q\leqq p^+$. Then for every m and for every $r<\aleph_0$*

$$\mathbf{M}(m,p,q,r)\to\mathbf{B}.$$

MILLER's theorem can be considered as a generalization of BERNSTEIN's theorem which states that if p is infinite, then every family $\mathfrak{F}$ ($\overline{\overline{\mathfrak{F}}}=p$, $p(\mathfrak{F})=p$) (without any further assumption for property $\mathbf{C}(q,r)$) possesses property $\mathbf{B}$,[4] i. e.:

THEOREM 2. $\mathbf{M}(p,p,q,r)\to\mathbf{B}$ *if $p\geqq\aleph_0$ for every q and r.*

MILLER's counterexample (1. 5) can be stated generally as follows:

THEOREM 3. $\mathbf{M}(2^p,p,2,p)\not\to\mathbf{B}$ *if p is infinite.*

Theorem 3 can be proved quite similarly as its special case for $p=\aleph_0$ cited in (1. 5) and therefore we omit the proof.

Theorem 2 shows that in the investigations concerning property $\mathbf{B}$ we may always suppose that $m\geqq p$, and Theorem 3 shows that if $m>p$, then to obtain positive results we have to suppose $r<p$.

We mention that without using (∗) we can not decide the following

PROBLEM 1. $\mathbf{M}(\aleph_1,\aleph_0,2,\aleph_0)\not\to\mathbf{B}$?

We can not prove without (∗) that every $\mathfrak{F}$ ($p(\mathfrak{F})=\aleph_0$, $\overline{\overline{\mathfrak{F}}}<2^{\aleph_0}$) possesses property $\mathbf{B}$.

[4] See [4].

It is obvious that the property $\mathbf{C}(q', r)$ is weaker than the property $\mathbf{C}(q, r)$, provided $q' > q$.

Now we prove:

(3.4) $$\mathbf{M}(2^p, p, q, 1) \nrightarrow \mathbf{B} \quad \textit{if} \quad q > 2^p, \quad p \geqq \aleph_0.$$

PROOF. Put $\mathfrak{F} = [S]^p$ where S is a set of power p. It is obvious that $\bar{\bar{\mathfrak{F}}} = 2^p$, $p(\mathfrak{F}) = p$, $\mathfrak{F}$ possesses property $\mathbf{C}(q, 1)$, but it does not possess property $\mathbf{B}$.

(3.4) shows that we have to suppose $q \leqq 2^p$, and since here we do not want to discuss the difficulties caused by the continuum problem, we are going to suppose $q \leqq p^+$.

It results that the best-possible generalization of Theorem 1 would be the following:

(o) $\mathbf{M}(m, p, q, r) \to \mathbf{B}$ *for every* m, *provided* $q \leqq p^+$, $p \geqq \aleph_0$, $r < p$.

We can prove this only with the stronger assumption $r^+ < p$ (see Theorem 4) or with the restriction that m is not too large (see Theorem 5). Both proofs use (∗).

The simplest unsolved problem here is

PROBLEM 2. $\mathbf{M}(\aleph_{\omega+1}, \aleph_1, 2, \aleph_0) \to \mathbf{B}$?

As to the symbol-II the problems are more ramified. First we have to discuss the case $m \leqq p$ which leads to some interesting result too. This will be done in Section 4. (3.2) shows that we have to suppose $s \leqq p$. In Section 4 we are going to prove that at least in the case $p \geqq \aleph_0$, $m \geqq p$ we may suppose $r^+ \leqq s$.

So the best-possible refinement of the conjecture (o) would be the following:

(oo) $\mathbf{M}(m, p, r) \to \mathbf{B}(r^+)$ *for every* $m \geqq p \geqq \aleph_0$, *provided* $r < p$.

Now we have to distinguish two cases:

(i) If r is finite, then (oo) is false. However, it is always true for $\aleph_0$ instead of r^+ and using (∗) corresponding to every m, p and r we can determine the least s (eventually finite) for which $\mathbf{M}(m, p, r) \to \mathbf{B}(s)$ is true. These results will be proved in Section 7. As a consequence of these results we prove the topological theorem mentioned in the introduction (in Section 8). There we state many conjectures which all would have been consequences of 2-compactness of the topological product of $\aleph_2$ Lindelöf spaces — now disproved — and which we can not disprove with our method.

(ii) If r is infinite, (oo) is very likely true, however, we can prove it — using (∗) — only with similar restrictions as in the case of symbol-I, namely we can prove that

$$\mathbf{M}(m, p, r) \to \mathbf{B}(r^+) \quad \text{for} \quad m \geqq p > r > \aleph_0,$$

provided m is not too large (see Theorem 7), and we can prove that

$$\mathbf{M}(m, p, r) \to \mathbf{B}(r^{++}) \quad \text{for every} \quad m \geqq p > r^+ > \aleph_0$$

(see Theorem 6).

The simplest unsolved problems here are

PROBLEM 3.

a) $\mathbf{M}(\aleph_{\omega+1}, \aleph_{\omega+1}, \aleph_0) \to \mathbf{B}(\aleph_1)$?

b) $\mathbf{M}(\aleph_{\omega+1}, \aleph_1, \aleph_0) \to \mathbf{B}(\aleph_1)$?

c) $\mathbf{M}(\aleph_{\omega+1}, \aleph_2, \aleph_1) \to \mathbf{B}(\aleph_2)$?

The results on (o) and (oo) will be proved in Section 6. All the positive results concerning the case $m > p$ will be proved with the method of MILLER's theorem, and the proof runs always by induction on m. That is why we need a generalization of the induction process used in [1]. This will be done in Section 5 and as a corollary of it we obtain all the positive theorems (Theorems 4—9) already mentioned.

In Section 9 we deal with the case of finite sets ($p < \aleph_0$) and with some questions related to property **B**.

4. The symbol-II in the cases $m \leqq p$ ($p \geqq \aleph_0$). The following theorems of A. TARSKI will play an important role in our investigations:

(*) LEMMA 1. *Let S be a set, $\mathfrak{F}$ a family such that $(\mathfrak{F}) \subseteq S$, $\bar{\bar{S}} = \aleph_\alpha$, $|\mathfrak{F}| \geqq \aleph_\beta$. Then*

a) *$\bar{\bar{\mathfrak{F}}} \leqq \aleph_\alpha$, provided $\mathfrak{F}$ possesses property $\mathbf{C}(\aleph_{\alpha+1}, \aleph_\beta)$ and $cf(\alpha) \neq cf(\beta)$.*

b) *$\bar{\bar{\mathfrak{F}}} \leqq \aleph_\alpha$, provided $\mathfrak{F}$ possesses property $\mathbf{C}(\aleph_{\alpha+1}, r)$ for an $r < \aleph_\beta$.*[5]

Note that in TARSKI's paper the theorems are proved under the stronger conditions that $\mathfrak{F}$ possesses the properties $\mathbf{C}(2, \aleph_\beta)$ and $\mathbf{C}(2, r)$, respectively, however, the proofs can be carried out in the same way for our case too.

LEMMA 2. *Let S be a set, $\mathfrak{F}$ a family such that $(\mathfrak{F}) \subseteq S$, $\bar{\bar{S}} = \aleph_\alpha$, $|\mathfrak{F}| \geqq r$ where r is finite. Then $\bar{\bar{\mathfrak{F}}} \leqq \aleph_\alpha$, provided $\mathfrak{F}$ possesses property $\mathbf{C}(\aleph_{\alpha+1}, r)$.*

Lemma 2 is a corollary of the fact that $\overline{\overline{[S]^r}} = \aleph_\alpha$ for every finite r. Note that the proof of Lemma 2 does not make use of (*).

(4. 1) $\mathbf{M}(m, p, r) \to \mathbf{B}(m^+)$ *for every* m, p, r.

$\mathbf{M}(m, p, r) \not\to \mathbf{B}(m)$ *if* $r > p$ *and* $p \geqq \aleph_0$ $(\aleph_0 \leqq m \leqq p)$.

[5] See [2], Theorem 5, I, p. 211 and Corollary 6, p. 213 for a) and b), respectively.

PROOF. The first statement is trivial, the second is to be seen quite similarly to (3. 3).

REMARK. *If* m *is finite,* $m \geqq 2$, *then* $\mathbf{M}(m, p, r) \to \mathbf{B}(m)$ *and* $\mathbf{M}(m, p, r) \not\to \mathbf{B}(m-1)$ *is true under the same conditions for* p, r *respectively.*

(4. 1) shows that we may always suppose $r \leqq p$.

(4. 2) $\mathbf{M}(m, \aleph_\alpha, r) \to \mathbf{B}(2)$ *for every* α *if* $m < \aleph_\alpha$ *and* $r < \aleph_\alpha$. *If* $r = \aleph_\alpha$, *then the same is true for* $m < \aleph_{cf(\alpha)}$.

PROOF. Let $\mathfrak{F}$ be a family $(p(\mathfrak{F}) = \aleph_\alpha,\ \overline{\overline{\mathfrak{F}}} = m)$ which possesses property $\mathbf{C}(2, r)$. It is obvious that the set $F - \bigcup_{F' \in \mathfrak{F},\, F' \neq F} F'$ is of power $\aleph_\alpha$ and so it is non-empty for an arbitrary $F \in \mathfrak{F}$. Let x_F be an element of this set and put $B = \{x_F\}_{F \in \mathfrak{F}}$. We have $\overline{\overline{B \cap F}} = 1$ for every $F \in \mathfrak{F}$, hence $\mathfrak{F}$ possesses property $\mathbf{B}(2)$.

(4. 3) $\mathbf{M}(\aleph_{cf(\alpha)}, \aleph_\alpha, \aleph_\alpha) \to \mathbf{B}(\aleph_{cf(\alpha)})$ *for every* α.

PROOF. Let $\mathfrak{F}$ be a family such that $p(\mathfrak{F}) = \aleph_\alpha$, $\overline{\overline{\mathfrak{F}}} = \aleph_{cf(\alpha)}$, and suppose that $\mathfrak{F}$ possesses property $\mathbf{C}(2, \aleph_\alpha)$. Let $\mathfrak{F} = \{F_\nu\}_{\nu < \omega_{cf(\alpha)}}$ be a well-ordering of $\mathfrak{F}$.

The set $F_\nu - \bigcup_{\mu < \nu} F_\mu$ is of power $\aleph_\alpha$ for every $\nu < \omega_{cf(\alpha)}$. Let x_ν be an element of it.

Put $B = \{x_\nu\}_{\nu < \omega_{cf(\alpha)}}$. It is obvious that $B \cap F_\nu \neq 0$ for every $\nu < \omega_{cf(\alpha)}$ and $\overline{\overline{B \cap F_\nu}} < \aleph_{cf(\alpha)}$ for every $\nu < \omega_{cf(\alpha)}$, since if $\nu' > \nu$, then $x_{\nu'} \notin F_\nu$. It follows that $\mathfrak{F}$ possesses property $\mathbf{B}(\aleph_{cf(\alpha)})$.

Now we show that (4. 2) is best-possible in "s", i. e.

(4. 4) $\mathbf{M}(\aleph_{cf(\alpha)}, \aleph_\alpha, \aleph_\alpha) \not\to \mathbf{B}(s)$ *if* $s < \aleph_{cf(\alpha)}$.

PROOF. We are going to suppose that α is of the second kind. If α is of the first kind, the statement can be proved quite similarly. Let S be a set of power $\aleph_\alpha$ and let $S = \{x_\beta\}_{\beta < \omega_\alpha}$ be a well-ordering of type ω_α of S. Let $\{\alpha_\nu\}_{\nu < \omega_{cf(\alpha)}}$ be a monotone increasing sequence of type $\omega_{cf(\alpha)}$ of ordinal numbers less than α cofinal with α. Put $S_\nu = \{x_\beta\}_{\beta < \omega_{\alpha_\nu}}$ for every $\nu < \omega_{cf(\alpha)}$. Obviously, one can define the sequences $\{F_\nu^1\}_{\nu < \omega_{cf(\alpha)}}$, $\{F_\nu^2\}_{\nu < \omega_{cf(\alpha)}}$ of type $\omega_{cf(\alpha)}$ of subsets of S in such a way that — if we put $\mathfrak{F} = \{F_\nu^1\}_{\nu < \omega_{cf(\alpha)}} \cup \cup \{F_\nu^2\}_{\nu < \omega_{cf(\alpha)}}$, then $\mathfrak{F}$ possesses property $\mathbf{C}(2, \aleph_\alpha)$ — and that the following conditions hold:

(1) $\overline{\overline{F_\nu^1}} = \overline{\overline{F_\nu^2}} = \aleph_\alpha$ for $\nu < \omega_{cf(\alpha)}$,

(2) $S_\nu \subseteq F_\nu^1$ for $\nu < \omega_{cf(\alpha)}$,

(3) $F_\nu^2 \cap F_{\nu'}^2 = 0$ for $\nu \neq \nu'$, $\nu, \nu' < \omega_{cf(\alpha)}$.

Then $p(\mathfrak{F}) = \aleph_\alpha$ by (1) and $\overline{\overline{\mathfrak{F}}} = \aleph_{cf(\alpha)}$. Suppose that the set B intersects every set F of the family $\mathfrak{F}$. Then $\overline{\overline{B}} \geqq \aleph_{cf(\alpha)}$ by (3), hence if $s < \aleph_{cf(\alpha)}$, there is a $B' \subseteq B$ such that $\overline{\overline{B'}} = s$ and there is a $\nu_0 < \omega_{cf(\alpha)}$ such that $B' \subseteq S_{\nu_0}$. But this means by (2) that $\overline{\overline{B \cap F^1_{\nu_0}}} \geqq s$ and thus $\mathfrak{F}$ does not possess property $\mathbf{B}(s)$.

It results from (4.1), (4.2), (4.3) and (4.4) that to complete the discussion of the case $m \leqq p$ $(p \geqq \aleph_0)$ we have to determine the value of symbol-II in the following cases:

A) $m = p$, $r < p$ $(p \geqq \aleph_0)$,

B) $m = \aleph_\beta$, $p = \aleph_\alpha$, $r = \aleph_\alpha$ where $cf(\alpha) < \beta \leqq \alpha$.

To obtain complete results we have to assume (*) in both cases. In the case A) there remains an unsolved problem even if we assume (*).

First we prove the following negative result concerning A):

(*) (4.5) $\mathbf{M}(\aleph_\alpha, \aleph_\alpha, r) \nrightarrow \mathbf{B}(r)$ *if* $r < \aleph_\alpha$.

(If r is finite, the assumption (*) can be omitted.)

PROOF. Let $\mathfrak{F}_1, \mathfrak{F}_2$ be families satisfying the following conditions:

(4.5.1) $$\overline{\overline{\mathfrak{F}_1}} = r^+, \quad \overline{\overline{\mathfrak{F}_2}} = \aleph_\alpha,$$

(4.5.2) $$p(\mathfrak{F}_1 \cup \mathfrak{F}_2) = \aleph_\alpha,$$

(4.5.3) $$F \cap F' = 0 \quad \text{for every pair } F, F' \in \mathfrak{F}_1 \cup \mathfrak{F}_2, \; F \neq F'.$$

Put $(\mathfrak{F}_1) = S_1$. By Zorn's lemma there exists a *maximal* system $\mathfrak{S}$ of subsets of S_1 satisfying the following conditions:

(4.5.4) $$\overline{\overline{X \cap F}} = 1 \quad \text{for every} \quad X \in \mathfrak{S}, F \in \mathfrak{F}_1,$$
$$\overline{\overline{X \cap Y}} < r \quad \text{for every pair} \quad X, Y \in \mathfrak{S}, \; X \neq Y.$$

From (4.5.1) and (4.5.4) we get

(4.5.5) $$\overline{\overline{X}} = r^+ \quad \text{for every} \quad X \in \mathfrak{S}$$

and using the maximality of $\mathfrak{S}$ we obtain:

(4.5.6) If the set B' intersects every set F of $\mathfrak{F}_1$, then there exists an element X_0 of $\mathfrak{S}$ such that $\overline{\overline{B' \cap X_0}} \geqq r$.

On the other hand, using Lemmas 1 and 2 we get from (4.5.1), (4.5.2) and (4.5.4) that

(4.5.7) $$\overline{\overline{\mathfrak{S}}} = \aleph_\alpha.$$

It follows that there exists a one-to-one mapping $h(X)$ which maps $\mathfrak{S}$ onto $\mathfrak{F}_2$. Put $\mathfrak{F}_2^* = \{h(X) \cup X\}_{X \in \mathfrak{S}}$ and define $\mathfrak{F}$ as follows:

$$\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2^*.$$

Since $r^+ \leqq \aleph_\alpha$ by the assumption, by (4.5.2) and (4.5.5) we have $p(\mathfrak{F}) = \aleph_\alpha$. By (4.5.3) and (4.5.4) $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$ and by (4.5.1) and (4.5.7) $\overline{\overline{\mathfrak{F}}} = \aleph_\alpha$.

We have to prove that $\mathfrak{F}$ does not possess property $\mathbf{B}(r)$. But if the set B intersects every set F of $\mathfrak{F}$, then it has a subset B' satisfying the condition of (4.5.6), hence $\overline{\overline{B \cap X_0}} \geqq r$ for an $X_0 \in \mathfrak{S}$ and therefore $\overline{\overline{B \cap F_0}} \geqq r$ for $F_0 = h(X_0) \cup X_0$, hence for an $F_0 \in \mathfrak{F}$. Thus $\mathfrak{F}$ does not possess property $\mathbf{B}(r)$.

REMARK. We have proved the following somewhat more general statement: *The family $\mathfrak{F}$ constructed above is such that each set which intersects every element of $\mathfrak{F}_1$ has to intersect an element of $\mathfrak{F}_2^*$ in at least r points.*

Now we need some preliminary definitions.

DEF. (4.6) Let $\mathfrak{F}$ be an arbitrary family and let S be a set such that $(\mathfrak{F}) \subseteq S$. For an arbitrary subset X of S and for an arbitrary cardinal number t we define the subfamily $\mathfrak{G}(X, t, \mathfrak{F})$ as follows:

$$\mathfrak{G}(X, t, \mathfrak{F}) = \{F : F \in \mathfrak{F} \wedge \overline{\overline{F \cap X}} \geqq t\}.$$

DEF. (4.7) Let $\mathfrak{F}$ and S have the same meaning as in (4.6). For an arbitrary $X \subseteq S$ we define the family $\mathfrak{F}|X$ as follows:

$$\mathfrak{F}|X = \{F \cap X\}_{F \in \mathfrak{F}}.$$

(Note that $\mathfrak{F}|X$ is not necessarily a subfamily of $\mathfrak{F}$.)

The following assertions are immediate consequences of the above definitions.

(4.8.1) *Let q, r be arbitrary. The families $\mathfrak{G}(X, t, \mathfrak{F})$ and $\mathfrak{F}|X$ possess property $\mathbf{C}(q, r)$, provided the same holds for $\mathfrak{F}$.*

(4.8.2) $|\mathfrak{G}(X, t, \mathfrak{F})|X| \geqq t$.

(4.8.3) *If the family $\mathfrak{F}$ possesses property $\mathbf{C}(q, r)$ and $t \geqq r$, then*

$$\overline{\overline{\mathfrak{G}(X, t, \mathfrak{F})}} \leqq q^- \cdot \overline{\overline{\mathfrak{G}(X, t, \mathfrak{F})|X}}.$$

Now we prove the following positive theorem concerning A):

(*) (4.9) *Suppose $r < \aleph_\alpha$. Then $\mathbf{M}(\aleph_\alpha, \aleph_\alpha, r) \to \mathbf{B}(r^+)$, provided the following condition does not hold:*

(v) *There exist ordinal numbers β, γ such that $\alpha = \beta + 1$, $r = \aleph_\gamma$, $cf(\beta) = cf(\gamma)$ and $\gamma < \beta$.*

(If r is finite, the assumption (*) can be omitted.)

PROOF. Let $\mathfrak{F}$ be a family ($p(\mathfrak{F}) = \aleph_\alpha$, $\overline{\overline{\mathfrak{F}}} = \aleph_\alpha$) which possesses property $\mathbf{C}(2, r)$. Put $S = (\mathfrak{F})$. Then $\overline{\overline{S}} = \aleph_\alpha$. Let $S = \{x_\nu\}_{\nu < \omega_\alpha}$ and $\mathfrak{F} = \{F_\mu\}_{\mu < \omega_\alpha}$ be well-orderings of type ω_α of the set S and of the family $\mathfrak{F}$, respectively. We may suppose that $r^+ < \aleph_\alpha$, for if not, then $r^+ = \aleph_\alpha$ is regular and the theorem follows from (4. 3), since the symbol-II is decreasing in r by (3. 1). Now we define a subsequence $\{x_{\nu_\varrho}\}_{\varrho<\omega_\alpha}$ of S by induction on ϱ as follows:

Let x_{ν_0} be an arbitrary element of F_0 and put $F_0=F_{\mu_0}$. Suppose that the elements x_{ν_σ} are already defined for every $\sigma<\varrho$, for a $\varrho<\omega_\alpha$. Put

(4. 9. 1) $$S_\varrho=\{x_{\nu_\sigma}\}_{\sigma<\varrho}, \quad \mathcal{G}_\varrho=\mathcal{G}(S_\varrho, r, \mathfrak{F}).$$

It is obvious that $\overline{\overline{S}}_\varrho \leqq \bar{\varrho} < \aleph_\alpha$ and by (4. 8. 3) $\overline{\overline{\mathcal{G}}}_\varrho \leqq \overline{\overline{\mathcal{G}_\varrho|S_\varrho}}$.

But $\mathcal{G}_\varrho|S_\varrho \subseteq \mathfrak{F}(S_\varrho)$, thus using (*) we get

(4. 9. 2) $$\overline{\overline{\mathcal{G}}}_\varrho<\aleph_\alpha \quad \text{except if} \quad \overline{\overline{S}}_\varrho^{\,+}=\bar{\varrho}^+=\aleph_\alpha.$$

Put $\bar{\varrho}=\aleph_\beta$ and suppose $\beta+1=\alpha$. Then $\overline{\overline{S}}_\varrho=\aleph_\beta$, $|\mathcal{G}_\varrho|S_\varrho|\geqq r$ (by 4. 8. 2). Using Lemmas 1 and 2 for the family $\mathcal{G}_\varrho|S_\varrho$ we get

(4. 9. 3) $$\overline{\overline{\mathcal{G}}}_\varrho \leqq \overline{\overline{\mathcal{G}_\varrho|S_\varrho}} \leqq \aleph_\beta<\aleph_\alpha \quad \text{except if} \quad r=\aleph_\gamma,\ cf(\beta)=cf(\gamma).$$

Thus it results from (4. 9. 2) and (4. 9. 3) and from the assumption that v) does not hold that $\overline{\overline{\mathcal{G}}}_\varrho<\aleph_\alpha$. Let μ_ϱ be the least ordinal number μ for which

$$S_\varrho \cap F_\mu=0.^{6}$$

It is obvious that $F_{\mu_\varrho}\notin\mathcal{G}_\varrho$, and so the set

$$F_{\mu_\varrho}-(\mathcal{G}_\varrho)=F_{\mu_\varrho}-\bigcup_{F_\mu\in\mathcal{G}_\varrho}(F_{\mu_\varrho}\cap F_\mu)$$

is of power $\aleph_\alpha$, since $\overline{\overline{F}}_{\mu_\varrho}=\aleph_\alpha$ and $\overline{\overline{\bigcup_{F_\mu\in\mathcal{G}_\varrho}(F_{\mu_\varrho}\cap F_\mu)}}\leqq r\cdot\bar{\varrho}<\aleph_\alpha$.

Thus there exists a ν such that $x_\nu\in F_{\mu_\varrho}-((\mathcal{G}_\varrho)\cup S_\varrho))$.

Let ν_ϱ be the least ν of this kind. Thus x_{ν_ϱ}, F_{μ_ϱ} are defined for every $\varrho<\omega_\alpha$ and it follows by induction on ϱ that

(4. 9. 4) $$x_{\mu_\varrho}\in F_{\mu_\varrho},\ x_{\nu_\varrho}\notin(\mathcal{G}_\varrho) \quad \text{and} \quad x_{\nu_\sigma}\neq x_{\nu_\varrho},\ \mu_\sigma\neq\mu_\varrho \quad \text{for every} \quad \sigma<\varrho<\omega_\alpha.$$

Put $B=\{x_{\nu_\varrho}\}_{\varrho<\omega_\alpha}$. Now we prove

(4. 9. 5) $$B\cap F_\mu\neq 0 \quad \textit{for every} \quad \mu<\omega_\alpha.$$

For if not, then there exists a least μ^0 of this kind, and by (4. 9. 4) there is a $\mu_\varrho>\mu^0$ in contradiction to the definition of F_{μ_ϱ}.

(4. 9. 6) $$\overline{\overline{B\cap F_\mu}}<r^+ \quad \textit{for every} \ \mu<\omega_\alpha.$$

[6] If such a μ does not exist, then we stop with the construction and obviously one can prove in the same way that S_ϱ assures property $\mathbf{B}(r^+)$ as we shall prove it later for B.

For if not, then there exists a $\mu^0 < \omega_\alpha$, a subset $B' \subset B$ and an $x_{\nu_{\varrho_0}} \in B$ such that $\overline{\overline{B'}} = r$, $B' + \{x_{\nu_{\varrho_0}}\} \subseteq F_{\mu_0}$ and $\sigma < \varrho_0$ for every $x_{\nu_\sigma} \in B'$, and this obviously contradicts (4. 9.4), since then $F_{\mu_0} \in G_{\varrho_0}$.

(4. 9. 5) and (4. 9. 6) just mean that the family $\mathfrak{F}$ possesses property $\mathbf{B}(r^+)$.

REMARK. As we have already mentioned in Problem 3a) — for a special case — we do not know whether $\mathbf{M}(\aleph_{\beta+1}, \aleph_{\beta+1}, \aleph_\gamma) \to \mathbf{B}(\aleph_{\gamma+1})$ is true or not if β and γ satisfy the assumption (v), i. e. if $cf(\beta) = cf(\gamma)$ and $\gamma < \beta$.

Now we need the following

(*) LEMMA 3. *Let S be a set, $\overline{\overline{S}} = \aleph_{\alpha+1}$, and suppose that $\aleph_\alpha$ is regular. Then there exists a system $\mathfrak{S}$ of subsets of S satisfying the following conditions:*

$p(\mathfrak{S}) = \aleph_\alpha$, $\mathfrak{S}$ possesses property $\mathbf{C}(2, \aleph_\alpha)$ and for an arbitrary $S' \subseteq S$ $(\overline{\overline{S'}} = \aleph_{\alpha+1})$ there exists an $X \in \mathfrak{S}$ such that $X \subseteq S'$.

Lemma 3 is a theorem of A. HAJNAL.[7]

Now turning to the case B) we are going to prove that *if $\aleph_\alpha$ is singular and $cf(\alpha) < \beta \leqq \alpha$, then the trivial result $\mathbf{M}(\aleph_\beta, \aleph_\alpha, \aleph_\alpha) \to \mathbf{B}(\aleph_{\beta+1})$ (see (4. 1)) is best-possible, i. e.*

(*) (4. 10) $$\mathbf{M}(\aleph_\beta, \aleph_\alpha, \aleph_\alpha) \not\to \mathbf{B}(\aleph_\beta) \quad \text{if} \quad cf(\alpha) < \beta \leqq \alpha.$$

We are going to prove this only for the case $\beta = \alpha$, the proof can be carried out similarly in the other cases too.[8]

Proof of the case $\beta = \alpha$. Let $\mathfrak{F}_1$ be a family satisfying the following conditions:

(4. 10. 1) $\overline{\overline{\mathfrak{F}_1}} = \aleph_{cf(\alpha)+1}$, $p(\mathfrak{F}_1) = \aleph_\alpha$ and $F \cap F' = 0$ for every $F, F' \in \mathfrak{F}_1$, $F \neq F'$.

$\aleph_{cf(\alpha)}$ being regular, we can apply Lemma 3 with $\mathfrak{F}_1$ instead of S and we obtain that there exists a system $\mathfrak{S}$ of subfamilies $\mathfrak{X}$ of $\mathfrak{F}_1$ satisfying the following conditions:

(4. 10. 2) $\overline{\overline{\mathfrak{S}}} = \aleph_{cf(\alpha)+1}$, $p(\mathfrak{S}) = \aleph_{cf(\alpha)}$, $\mathfrak{S}$ possesses property $\mathbf{C}(2, \aleph_{cf(\alpha)})$ and if $\mathfrak{F}'$ is a subfamily of $\mathfrak{F}_1$ such that $\overline{\overline{\mathfrak{F}'}} = \aleph_{cf(\alpha)+1}$, then there exists an $\mathfrak{X} \in \mathfrak{S}$ for which $\mathfrak{X} \subseteq \mathfrak{F}'$.

Let $\mathfrak{S} = \{\mathfrak{X}_\mu\}_{\mu < \omega_{cf(\alpha)+1}}$ and $\mathfrak{F}_1 = \{F_\tau\}_{\tau < \omega_{cf(\alpha)+1}}$ be well-orderings of type $\omega_{cf(\alpha)+1}$ of $\mathfrak{S}$ and $\mathfrak{F}_1$, respectively. Let further $\{\alpha_\nu\}_{\nu < \omega_{cf(\alpha)}}$ be a monotone increasing sequence of type $\omega_{cf(\alpha)}$ of ordinal numbers less than α cofinal with α.

[7] See [6], Theorem 9.

[8] We mention that if $cf(\alpha) < cf(\beta)$ (especially, if β is of the first kind), then the theorem is easy and can be proved without using (*). But for the cases $cf(\beta) \geqq cf(\alpha)$ we have to use the same complicated proof as for the case $\beta = \alpha$. It is possible that a simpler proof can be constructed in this case too, but we were unsuccessful in doing this.

By (4.10.2) $\mathfrak{X}_\mu \subseteq \mathfrak{F}_1$ and $\overline{\overline{\mathfrak{X}_\mu}} = \aleph_{cf(\alpha)}$ for every $\mu < \omega_{cf(\alpha)+1}$. Let $\mathfrak{X}_\mu = \{F_\nu^\mu\}_{\nu < \omega_{cf(\alpha)}}$ be a well-ordering of type $\omega_{cf(\alpha)}$ of $\mathfrak{X}_\mu$.

The set F_ν^μ — being an element of $\mathfrak{F}_1$ — is of power $\aleph_\alpha$, and so it can be split into the sum of $\aleph_\alpha$ disjoint subsets of power $\aleph_{\alpha_\nu}$, that means: there exists a sequence $\{F_\nu^\mu(\gamma)\}_{\gamma < \omega_\alpha}$ of type ω_α of subsets of F_ν^μ satisfying the following conditions:

(4.10.3) $\overline{\overline{F_\nu^\mu(\gamma)}} = \aleph_{\alpha_\nu}$ for every $\gamma < \omega_\alpha$, $F_\nu^\mu(\gamma_1) \cap F_\nu^\mu(\gamma_2) = 0$ for every $\gamma_1, \gamma_2 < \omega_\alpha$, $\gamma_1 \neq \gamma_2$, and $F_\nu^\mu = \bigcup_{\gamma < \omega_\alpha} F_\nu^\mu(\gamma)$ where $\mu < \omega_{cf(\alpha)+1}$, $\nu < \omega_{cf(\alpha)}$ are arbitrary.

Now, corresponding to every $\mu < \omega_{cf(\alpha)+1}$ we define a family $\mathfrak{F}_{2,\mu}$ as follows. First put $F^\mu(\gamma) = \bigcup_{\nu < \omega_{cf(\alpha)}} F_\nu^\mu(\gamma)$, and then put $\mathfrak{F}_{2,\mu} = \{F^\mu(\gamma)\}_{\gamma < \omega_\alpha}$.

We have, for every $\mu < \omega_{cf(\alpha)+1}$,

(4.10.4) $\overline{\overline{\mathfrak{F}_{2,\mu}}} = \aleph_\alpha$, $F^\mu(\gamma_1) \cap F^\mu(\gamma_2) = 0$ for $\gamma_1, \gamma_2 < \omega_\alpha, \gamma_1 \neq \gamma_2$, $p(\mathfrak{F}_{2,\mu}) = \aleph_\alpha$ and $(\mathfrak{F}_{2,\mu}) = (\mathfrak{X}_\mu)$.

In fact, the second statement follows from (4.10.1) and (4.10.3), the first one is a corollary of it, while the third and the fourth ones are consequences of (4.10.3), since

$$\overline{\overline{F^\mu(\gamma)}} = \sum_{\nu < \omega_{cf(\alpha)}} \overline{\overline{F_\nu^\mu(\gamma)}} = \sum_{\nu < \omega_{cf(\alpha)}} \aleph_{\alpha_\nu} = \aleph_\alpha \text{ for every } \mu < \omega_{cf(\alpha)+1}, \gamma < \omega_\alpha.$$

Now we put

$$\mathfrak{F} = \mathfrak{F}_1 \cup \bigcup_{\mu < \omega_{cf(\alpha)+1}} \mathfrak{F}_{2,\mu}.$$

We have

$$p(\mathfrak{F}) = \aleph_\alpha, \tag{4.10.5}$$

since $p(\mathfrak{F}_1) = p(\mathfrak{F}_{2,\mu}) = \aleph_\alpha$ for $\mu < \omega_{cf(\alpha)+1}$ by (4.10.1) and (4.10.4).

Taking into consideration that $\aleph_\alpha$ is singular and therefore $cf(\alpha)+1 < \alpha$, we get from (4.10.1) and (4.10.4)

$$\overline{\overline{\mathfrak{F}}} = \overline{\overline{\mathfrak{F}_1}} + \sum_{\mu < \omega_{cf(\alpha)+1}} \overline{\overline{\mathfrak{F}_{2,\mu}}} = \aleph_{cf(\alpha)+1} + \aleph_{cf(\alpha)+1} \cdot \aleph_\alpha = \aleph_\alpha. \tag{4.10.6}$$

We are going to prove that

(4.10.7) $\mathfrak{F}$ possesses property $\mathbf{C}(2, \aleph_\alpha)$.

Let F, F' be two elements of $\mathfrak{F}$ such that $F \neq F'$.

To see that $\overline{\overline{F \cap F'}} < \aleph_\alpha$, we distinguish four cases: (i) $F, F' \in \mathfrak{F}_1$, (ii) $F \in \mathfrak{F}_1$, $F' \in \mathfrak{F}_{2,\mu}$, (iii) $F \in \mathfrak{F}_{2,\mu}$, $F' \in \mathfrak{F}_{2,\mu}$, (iiii) $F \in \mathfrak{F}_{2,\mu}$, $F' \in \mathfrak{F}_{2,\mu'}$ for some $\mu \neq \mu'$.

In the cases (i) and (iii) F and F' are disjoint by (4.10.1) and (4.10.4), respectively. If (ii) holds, then — by (4.10.3) — either $F \cap F' = 0$ if $F \notin \mathfrak{X}_\mu$, or if $F \in \mathfrak{X}_\mu$, then $F = F_\nu^\mu$ for a $\nu < \omega_{cf(\alpha)}$ and $F' = F^\mu(\gamma)$ for a $\gamma < \omega_\alpha$ and $\overline{\overline{F \cap F'}} = \overline{\overline{F_\nu^\mu(\gamma)}} = \aleph_{\alpha_\nu} < \aleph_\alpha$.

Suppose now that (iiii) holds. By (4.10.2) we have $\overline{\overline{\mathfrak{X}_\mu \cap \mathfrak{X}_{\mu'}}} < \aleph_{cf(\alpha)}$, since $\mu \neq \mu'$. It is obvious that $(\mathfrak{F}) = \bigcup_{\tau < \omega_{cf(\alpha)+1}} F_\tau$, and so $F \cap F' = \bigcup_{\tau < \omega_{cf(\alpha)+1}} (F_\tau \cap (F \cap F'))$ but either $F_\tau \cap F$ or $F_\tau \cap F'$ is empty if $F_\tau \notin \mathfrak{X}_\mu \cap \mathfrak{X}_{\mu'}$, hence

$$F \cap F' = \bigcup_{F_\tau \in \mathfrak{X}_\mu \cap \mathfrak{X}_{\mu'}} (F_\tau \cap (F \cap F')).$$

Taking into consideration that by (4.10.3) $\overline{\overline{F_\tau \cap F}} < \aleph_\alpha$, it results that $\overline{\overline{F \cap F'}} < \aleph_\alpha$ in this case too.

It remains to prove that

(4.10.8) $\mathfrak{F}$ does not possess property $\mathbf{B}(\aleph_\alpha)$.

Let B be a set such that $\overline{\overline{B \cap F}} < \aleph_\alpha$ for every $F \in \mathfrak{F}$. Then, especially, corresponding to every $\tau < \omega_{cf(\alpha)+1}$ there exists a subscript $\nu(\tau) < \omega_{cf(\alpha)}$ such that

$$\overline{\overline{B \cap F_\tau}} < \aleph_{\alpha_{\nu(\tau)}}.$$

It results that there exists a subfamily $\mathfrak{F}'$ of $\mathfrak{F}_1$ and an ordinal number $\nu_0 < \omega_{cf(\alpha)}$ such that $\overline{\overline{\mathfrak{F}'}} = \aleph_{cf(\alpha)+1}$ and $\overline{\overline{B \cap F_\tau}} < \aleph_{\alpha_{\nu_0}}$ for every $F_\tau \in \mathfrak{F}'$. But, by (4.10.2), then there exists a $\mu_0 < \omega_{cf(\alpha)+1}$ such that $\mathfrak{X}_{\mu_0} \subseteq \mathfrak{F}'$. Thus we have $\overline{\overline{F_\nu^{\mu_0} \cap B}} < \aleph_{\alpha_{\nu_0}}$ for every $\nu < \omega_{cf(\alpha)}$, and so $\overline{\overline{B \cap (\mathfrak{X}_{\mu_0})}} \leq \aleph_{cf(\alpha)} \cdot \aleph_{\alpha_{\nu_0}} < \aleph_\alpha$. But by (4.10.4) $\mathfrak{F}_{2,\mu_0}$ consists of $\aleph_\alpha$ disjoint subsets of $(\mathfrak{X}_{\mu_0})$, consequently there is an $F \in \mathfrak{F}_{2,\mu_0} \subseteq \mathfrak{F}$ such that

$$B \cap F = 0.$$

Thus by (4.10.5)—(4.10.8) the case $\beta = \alpha$ of (4.10) is proved.

From (4.3) and (4.10) we obtain the following

(*) COROLLARY. *Suppose p is infinite. Then $\mathbf{M}(p, p, p) \to \mathbf{B}(p)$ holds if and only if p is a regular cardinal number.*

This should be compared with BERNSTEIN's theorem cited as Theorem 2 in Section 3.

REMARK. After (3.1) we have stated without proof that $\mathbf{M}(m, p, r) \to \mathbf{B}(s)$ is not monotone increasing in p. This can be seen e. g. by the following examples:

$\mathbf{M}(\aleph_1, \aleph_0, \aleph_2) \to \mathbf{B}(\aleph_1)$ holds by (3.2) but

$\mathbf{M}(\aleph_1, \aleph_1, \aleph_2) \not\to \mathbf{B}(\aleph_1)$ by (4.1); or

$\mathbf{M}(\aleph_2, \aleph_0, \aleph_1) \to \mathbf{B}(\aleph_1)$ by (3.2) but

(*) $\mathbf{M}(\aleph_2, \aleph_1, \aleph_1) \not\to \mathbf{B}(\aleph_1)$ by Theorem 3 and (3.2) and

(*) $\mathbf{M}(\aleph_2, \aleph_2, \aleph_1) \not\to \mathbf{B}(\aleph_1)$ by (4.5).

However, every example which disproves the monotonicity in question is such that $s>p$. Under the condition $s \leqq p$ — and these are the only genuine cases — the monotonicity seems to hold. Suppose namely that $\mathbf{M}(m,p,r)\to\mathbf{B}(s)$ is true, $s\leqq p$, and for the sake of simplicity suppose further that $m,p,r,s\geqq\aleph_0$, and suppose $(*)$.

Distinguish three cases: (i) $p<r$, (ii) $p=r$, (iii) $p>r$. If (i) holds, then by (3.1) and (4.1) $m^+\leqq s$, hence again by (3.1) and (4.1) $\mathbf{M}(m,p',r)\to\mathbf{B}(s)$ is true for every $p'>p$.

If (ii) holds, then $m\leqq p$ by Theorem 3 and by (3.2), and so $p'>m,r$ for every $p'>p$, hence $\mathbf{M}(m,p',r)\to\mathbf{B}(s)$ is true by (4.2).

If (iii) holds, then the implication is again trivial if $m\leqq p$, and if $m>p$, then by Theorem 6 which will be proved in Section 6 $\mathbf{M}(m,p',r)\to\mathbf{B}(p^+)$ is true for every $p'>p$.

By a slight modification of the proof of Theorem 6 one can obtain the following theorem:

$(*)$ *If $\mathfrak{F}$ is a family, $p(\mathfrak{F})=p'$, $\overline{\overline{\mathfrak{F}}}=m$ and $\mathfrak{F}$ possesses property $\mathbf{C}(2,r)$, then there exists a set B such that $\overline{\overline{B\cap F}}=p$ for every $F\in\mathfrak{F}$, provided that the above-mentioned inequalities hold for the cardinal numbers in question.*

Put $\mathfrak{F}'=\{B\cap F\}_{F\in\mathfrak{F}}$. It is obvious that $\mathfrak{F}$ possesses property $\mathbf{B}(s)$, provided the same holds for $\mathfrak{F}'$, but $p(\mathfrak{F}')=p$, $\overline{\overline{\mathfrak{F}'}}\leqq m$ and $\mathfrak{F}'$ possesses property $\mathbf{C}(2,r)$, hence $\mathbf{M}(m,p,r)\to\mathbf{B}(s)$ implies $\mathbf{M}(m,p',r)\to\mathbf{B}(s)$ in this case too.

It is possible that one can find a simpler proof for the monotonicity which does not use the hypothesis $(*)$, but we were unsuccessful in doing this.

5. Generalization of Miller's inductive construction. Let $(\mathfrak{F})$ be a family and S a set, $(\mathfrak{F})\subseteq S$.

DEF. (5.1) Let $\mathfrak{F}'$ be a subfamily of $\mathfrak{F}$ and put $S'=(\mathfrak{F}')$. $\mathfrak{F}'$ is said to be closed in $\mathfrak{F}$ with respect to the cardinal number t (or briefly t-closed in $\mathfrak{F}$) if $F\in\mathfrak{F}$ and $\overline{\overline{F\cap S'}}\geqq t$ implies that $F\in\mathfrak{F}'$.

It is obvious that if $\mathfrak{F}'$ is an arbitrary subfamily of $\mathfrak{F}$, then (the intersection of any number of t-closed subfamilies being t-closed) for every t there exists a minimal t-closed subfamily of $\mathfrak{F}$ containing $\mathfrak{F}'$. However, we need concrete constructions for t-closed families containing $\mathfrak{F}'$.

DEF. (5.2) We define the t, ε closure of $\mathfrak{F}'$ in $\mathfrak{F}$: $\mathrm{Clos}(\mathfrak{F}',\mathfrak{F},t,\varepsilon)$ for every ε. First we define a sequence $\{\mathfrak{G}_\nu\}_{\nu<\omega_\varepsilon}$ of type ω_ε of subfamilies of $\mathfrak{F}$ by induction on ν as follows:

Put $\mathfrak{G}_0=\mathfrak{F}'$ and $S_0=(\mathfrak{G}_0)$. Suppose that the families $\mathfrak{G}_\mu$ as well as the sets S_μ are already defined for every $\mu<\nu$ for a $\nu<\omega_\varepsilon$.

Put $S_\nu^* = \bigcup_{\mu<\nu} S_\mu$, $\mathfrak{G}_\nu = \mathfrak{G}(S_\nu^*, t, \mathfrak{F})$ (where $\mathfrak{G}$ is the function defined in (4.6)), and $S_\nu = (\mathfrak{G}_\nu)$.

Thus $\mathfrak{G}_\nu$ is defined for every $\nu < \omega_\varepsilon$. Now we put

$$\text{Clos}(\mathfrak{F}', \mathfrak{F}, t, \varepsilon) = \bigcup_{\nu<\omega_\varepsilon} \mathfrak{G}_\nu.$$

As an immediate consequence of the definition we get that

$$(\text{Clos}(\mathfrak{F}', \mathfrak{F}, t, \varepsilon)) = \bigcup_{\nu<\omega_\varepsilon} S_\nu \quad \text{and} \quad \mathfrak{F}' \subseteq \text{Clos}(\mathfrak{F}', \mathfrak{F}, t, \varepsilon).$$

We have:

(5.3) Clos $(\mathfrak{F}', \mathfrak{F}, t, \varepsilon)$ *is t-closed for every* $t < \aleph_{cf(\varepsilon)}$.[9]

PROOF. Let F be an element of $\mathfrak{F}$ such that $\overline{\overline{F \cap (\text{Clos}(\mathfrak{F}', \mathfrak{F}, t, \varepsilon))}} \geqq t$. Then $\overline{\overline{F \cap S_{\nu_0}^*}} \geqq t$ for a suitable $\nu_0 < \omega_\varepsilon$ and thus $F \in \mathfrak{G}_{\nu_0} \subseteq \text{Clos}(\mathfrak{F}', \mathfrak{F}, t, \varepsilon)$.

In what follows in this section let $\mathfrak{F}$ be a fixed family, $(\mathfrak{F}) = S$. Suppose that $p(\mathfrak{F}) = p$, $\overline{\overline{\mathfrak{F}}} = m$, $\mathfrak{F}$ possesses property $\mathbf{C}(q, r)$, where the cardinal numbers m, p, q, r, s and t satisfy the following inequalities:

$$(^\circ) \qquad m > p,\ p \geqq \aleph_0,\ 2 \leqq q \leqq p^+,\ r < p,\ r^+ \leqq s \leqq p,\ r \leqq t < p.$$

Every statement proved in this section depends on the assumption $(^\circ)$. We are going to use the notations $p = \aleph_\alpha$, $m = \aleph_\beta$, $r = \aleph_\gamma$, $s = \aleph_\delta$ alternatively (provided r and s are infinite).

DEF. (5.4) Let $\varepsilon(t)$ denote the index of the least $\aleph$ greater than t. ($\varepsilon(t) = 0$ if t is finite and $\aleph_{\varepsilon(t)} = t^+$ if t is infinite.) This means that $\aleph_{\varepsilon(t)}$ is always regular. Put briefly Clos $(\mathfrak{F}', t)$ for Clos $(\mathfrak{F}', \mathfrak{F}, t, \varepsilon(t))$.

We need the following

(∗) LEMMA 4. *Let $\mathfrak{F}'$ be a subfamily of $\mathfrak{F}$, $\overline{\overline{\mathfrak{F}'}} = m' \geqq p$. Then $\overline{\overline{\text{Clos}(\mathfrak{F}', t)}} = m'$, provided one of the following conditions (α) and ($\alpha\alpha$) holds:*

(α) *$r = t$ and the following condition does not hold:*

(vv) *There exist ordinal numbers β' and γ such that $m' = \aleph_{\beta'}$, $r = \aleph_\gamma$ and $cf(\beta') = cf(\gamma)$.*

($\alpha\alpha$) *$r < t$.*

(Note that in case r is finite, the hypothesis (∗) can be omitted.)

PROOF. Let $\mathfrak{G}_\nu$ denote the families defined in (5.2) corresponding to the given $\mathfrak{F}', t$ and $\varepsilon(t)$. First we are going to prove by induction on ν that

(1) $\overline{\overline{\mathfrak{G}_\nu}} = m'$ and $\overline{\overline{S_\nu}} \leqq m'$ for every $\nu < \omega_{\varepsilon(t)}$.

[9] It would be easy to see that (5.3) holds under more general conditions too, but we do not need this. E.g., it is true that for every t either Clos $(\mathfrak{F}', \mathfrak{F}, t, 0)$ or Clos $(\mathfrak{F}', \mathfrak{F}, t, 1)$ is t-closed.

This is true for $\nu=0$, since $\overline{\overline{\mathcal{G}_0}}=\overline{\overline{\mathfrak{F}'}}=m'$ by the assumption and $\overline{\overline{S_0}}=\overline{\overline{(\mathcal{G}_0)}}\leqq p\cdot m'=m'$. Suppose that the theorem is proved for every $\mu<\nu$ where $\nu<\omega_{\varepsilon(t)}$.

Then $\overline{\overline{S_\nu^*}}\leqq\sum_{\mu<\nu}m'=m'\overline{\nu}$. But by (5.4) $\overline{\nu}<\aleph_0\cdot t$ and therefore $\overline{\overline{S_\nu^*}}\leqq m'\cdot\aleph_0\cdot t=m'$, since $t<p\leqq m'$.

Now we obtain from (4.8.1), (4.8.2), (4.8.3) and (5.2) that $\overline{\overline{\mathcal{G}_\nu}}\leqq q^-\cdot\overline{\overline{\mathcal{G}_\nu|S_\nu^*}}$, $\mathcal{G}_\nu|S_\nu^*$ possesses property $\mathbf{C}(q,r)$ and $|\mathcal{G}_\nu|S_\nu^*|\geqq t$. On the other hand, we have $q\leqq(m')^+$, since $q\leqq p^+$ and $p\leqq m'$.

Hence by Lemmas 1 and 2 each of the conditions (α) and ($\alpha\alpha$) implies that $\overline{\overline{\mathcal{G}_\nu|S_\nu^*}}\leqq m'$. Consequently, we have $\overline{\overline{\mathcal{G}_\nu}}\leqq p\cdot m'=m'$, since $q^-\leqq p$ if $q\leqq p^+$. Thus $\overline{\overline{\mathcal{G}_\nu}}=m'$, since $\mathcal{G}_\nu$ contains $\mathcal{G}_0$, and similarly as for the case $\nu=0$ we obtain that $\overline{\overline{S_\nu}}\leqq m'$, and (1) is proved.

Using that $t<p$, $p\geqq\aleph_0$ implies $\aleph_{\varepsilon(t)}\leqq p$, we get from (1)

$$m'\leqq\overline{\overline{\mathrm{Clos}\,(\mathfrak{F}',t)}}\leqq\sum_{\nu<\omega_{\varepsilon(t)}}m'\leqq m'\cdot p=m'$$

and Lemma 4 is proved.

Let now $\mathfrak{F}=\{F_\varrho\}_{\varrho<\omega_\beta}$ be a well-ordering of type ω_β of the family $\mathfrak{F}$.

Now we are going to define the sequences $\{\mathfrak{F}'_\sigma(t)\}_{\sigma<\varphi}$, $\{\mathfrak{F}_\sigma(t)\}_{\sigma<\varphi}$ of type φ of subfamilies of $\mathfrak{F}$ as well as the sequence $\{S_\sigma(t)\}_{\sigma<\varphi}$ of subsets of S for a $\varphi\leqq\omega_\beta$ by induction on σ as follows:

DEF. (5.5) Put $\mathfrak{F}'_0(t)=\{F_\varrho\}_{\varrho<\omega_\alpha}$, $\mathfrak{F}_0(t)=\mathrm{Clos}\,(\mathfrak{F}'_0(t),t)$, $S_0(t)=(\mathfrak{F}_0(t))$. Suppose that the families $\mathfrak{F}'_{\sigma'}(t)$, $\mathfrak{F}_{\sigma'}(t)$ and the sets $S_{\sigma'}(t)$ are already defined for every $\sigma'<\sigma$. Put

$$\mathfrak{F}^*_\sigma(t)=\bigcup_{\sigma'<\sigma}\mathfrak{F}_{\sigma'}(t),\quad S^*_\sigma(t)=\bigcup_{\sigma'<\sigma}S_{\sigma'}(t).$$

If there exists an index $\varrho<\omega_\beta$ such that $F_\varrho\notin\mathfrak{F}^*_\sigma(t)$, then put $\varrho_\sigma=\varrho$ for the least ϱ of this kind, if not, then put $\sigma=\varphi$.

If ϱ_σ exists, then put

$$\mathfrak{F}'_\sigma(t)=\mathfrak{F}^*_\sigma(t)\cup\{\mathfrak{F}_{\varrho_\sigma}\},\quad\mathfrak{F}_\sigma(t)=\mathrm{Clos}\,(\mathfrak{F}'_\sigma(t),t),\quad S_\sigma(t)=(\mathfrak{F}_\sigma(t)).$$

Finally, if ϱ_σ is defined for every $\sigma<\omega_\beta$, then put $\varphi=\omega_\beta$.

(5.6) As an immediate consequence of the definition we obtain the following results:

(5.6.1) $$\mathfrak{F}=\bigcup_{\sigma<\varphi}\mathfrak{F}_\sigma(t),$$

(5.6.2) $\mathfrak{F}^*_{\sigma'}(t)\subset\mathfrak{F}'_{\sigma'}(t)\subseteq\mathfrak{F}_{\sigma'}(t)\subseteq\mathfrak{F}^*_\sigma(t)$ *for every* $\sigma'<\sigma<\varphi$,

(5.6.3) $S_{\sigma'}(t)\subseteq S^*_\sigma(t)\subseteq S_\sigma(t)$ *for every* $\sigma'<\sigma<\varphi$,

(5.6.4) $\mathfrak{F}_\sigma(t)$ *is t-closed in* $\mathfrak{F}$ *for every* $\sigma<\varphi$ *by* (5.3) *and* (5.4).

DEF. (5.7) Put $\mathcal{H}_\sigma(t) = \mathcal{F}_\sigma(t) - \mathcal{F}^*_\sigma(t)$ for every $\sigma < \varphi$.

By (5.5) and (5.6.1) we have

(5.7.1) $$\mathcal{F} = \bigcup_{\sigma<\varphi} \mathcal{H}_\sigma(t),$$

and by (5.6.2)

(5.7.2) $$\mathcal{H}_\sigma(t) \cap \mathcal{H}_{\sigma'}(t) = 0 \quad \text{for every} \quad \sigma' < \sigma < \varphi.$$

Now we prove the following lemma:

(5.8) *Suppose that* $F_\varrho \in \mathcal{H}_\sigma(t)$ *for some* $\varrho < \omega_\beta$, $\sigma < \varphi$. *Then*

(β) $\overline{\overline{F_\varrho \cap S^*_\sigma(t)}} \leqq t$, *and if* t *is finite, then*

($\beta\beta$) $\overline{\overline{F_\varrho \cap S^*_\sigma(t)}} < t$.

PROOF. First of all — $\mathcal{F}_{\sigma'}(t)$ being t-closed by (5.6.4) — we may suppose $\overline{\overline{F_\varrho \cap S_{\sigma'}(t)}} < t$ for every $\sigma' < \sigma$, for if not, then by the definition (5.1) F_ϱ belongs to $\mathcal{F}_{\sigma'}(t)$ in contradiction to (5.7).

We distinguish two cases: (i) $\sigma = \sigma_1 + 1$ for a $\sigma_1 < \sigma$, (ii) σ is of the second kind.

(i) By (5.5) and (5.6.3) we have $S^*_\sigma(t) = S_{\sigma_1}(t)$, hence ($\beta\beta$) holds for every t.

(ii) Let ω_τ be the least ordinal number cofinal with σ and let $\{\sigma_\eta\}_{\eta<\omega_\tau}$ be a monotone increasing sequence of ordinal numbers less than σ of type ω_τ cofinal with σ. We distinguish again two cases: (j) $\aleph_\tau \leqq t$, (jj) $\aleph_\tau > t$.

(j) We have by (5.5) and (5.6.3)

$$S^*_\sigma(t) = \bigcup_{\sigma'<\sigma} S_{\sigma'}(t) = \bigcup_{\eta<\omega_\tau} S_{\sigma_\eta}(t).$$

Hence $\overline{\overline{F_\varrho \cap S^*_\sigma(t)}} \leqq \sum_{\eta<\omega_\tau} \overline{\overline{S_{\sigma_\eta} \cap F_\varrho}} \leqq t \cdot \aleph_\tau = t$ and thus (β) holds.

(jj) Using again $S^*_\sigma(t) = \bigcup_{\eta<\omega_\tau} S_{\sigma_\eta}(t)$, we obtain that ($\beta\beta$) holds, for if not, then $F_\varrho \cap S^*_\sigma(t)$ contains a subset of power t which — $\aleph_\tau$ being regular — is contained already in a set $S_{\sigma_{\eta_0}}$ for an $\eta_0 < \tau$.

If t is finite, then either (i) or (jj) holds for it, and therefore if t is finite, then ($\beta\beta$) is true.

DEF. (5.9) By (5.7.1) and (5.7.2) corresponding to every $\varrho < \omega_\beta$ there exists exactly one $\sigma < \varphi$ such that $F_\varrho \in \mathcal{H}_\sigma(t)$. Put $\tilde{F}_\varrho = F_\varrho - S^*_\sigma(t)$ for this σ and put further $\tilde{\mathcal{H}}_\sigma(t) = \{\tilde{F}_\varrho\}_{F_\varrho \in \mathcal{H}_\sigma(t)}$. Put finally $\tilde{S}_\sigma(t) = (\tilde{\mathcal{H}}_\sigma(t))$.

We need the following results:

It results from the assumption $p(\mathcal{F}) = p > t$ by (5.8) and (5.9) that (5.10.1) $p(\tilde{\mathcal{H}}_\sigma(t)) = p$ *for every* $\sigma < \varphi$, and it is obvious from (5.9) that (5.10.2) *the family* $\tilde{\mathcal{H}}_\sigma(t)$ *possesses property* $\mathbf{C}(q, r)$ *for every* $\sigma < \varphi$.

(5. 10. 3) *Suppose* $F_\varrho \in \mathfrak{H}_\sigma(t)$. *Then*

(γ) $\overline{\overline{F_\varrho \cap \bigcup_{\sigma'<\sigma} \tilde{S}_{\sigma'}(t)}} \leqq t$ *and the equality is excluded if* t *is finite, and*

($\gamma\gamma$) $F_\varrho \cap \tilde{S}_{\sigma''}(t) = 0$ *for every* $\sigma'' > \sigma$.

PROOF. (γ) By the definitions (5. 7) and (5. 9) $\tilde{S}_{\sigma'}(t) \subseteq S_{\sigma'}(t) \subseteq S^*_\sigma(t)$ for every $\sigma' < \sigma$, hence by (5. 8) we get $\overline{\overline{F_\varrho \cap \bigcup_{\sigma'<\sigma} \tilde{S}_{\sigma'}(t)}} \leqq \overline{\overline{F_\varrho \cap S^*_\sigma(t)}} \leqq t$ (or $< t$ if t is finite).

($\gamma\gamma$) It is enough to see that $F_\varrho \cap \tilde{F}_{\varrho''} = 0$ for every $\tilde{F}_{\varrho''} \in \tilde{\mathfrak{H}}_{\sigma''}(t)$. But $F_\varrho \subseteq S_\sigma(t) \subseteq S^*_{\sigma''}(t)$, and so by (5. 9) $\tilde{F}_{\varrho''} \cap F_\varrho \subseteq \tilde{F}''_\varrho \cap S^*_{\sigma''} = 0$.

Now we prove the following

LEMMA 5. *Suppose that the families* $\tilde{\mathfrak{H}}_\sigma(t)$ *possess property* $\mathbf{B}(s)$ *for every* $\sigma < \varphi$. *Then the family* $\mathfrak{F}$ *possesses property* $\mathbf{B}(t^+ + s)$, *and if* t *is finite, then it possesses property* $\mathbf{B}((t-1)+s)$ *too.*

PROOF. By the assumption for every $\sigma < \varphi$ there exists a set B_σ such that $B_\sigma \subseteq \tilde{S}_\sigma(t)$ and $1 \leqq \overline{\overline{B_\sigma \cap \tilde{F}_\varrho}} < s$ for every $\tilde{F}_\varrho \in \tilde{\mathfrak{H}}_\sigma(t)$.

Put $B = \bigcup_{\sigma<\varphi} B_\sigma$. By (5. 7. 1) for every $\varrho < \omega_\beta$ there exists a $\sigma < \varphi$ such that $F_\varrho \in \mathfrak{H}_\sigma(t)$. Then $\tilde{F}_\varrho \in \tilde{\mathfrak{H}}_\sigma(t)$, $\tilde{F}_\varrho \subseteq F_\varrho$, by (5. 9), and B_σ intersects $\tilde{F}_\varrho$ by the assumption, hence we get

$$B \cap F_\varrho \neq 0 \quad \text{for every} \quad \varrho < \omega_\beta. \tag{1}$$

Now we are going to prove that

$$\overline{\overline{B \cap F_\varrho}} < t^+ + s \quad \text{for every} \quad \varrho < \omega_\beta. \tag{2}$$

Let now σ_ϱ be the uniquely determined ordinal number for which $F_\varrho \in \mathfrak{H}_{\sigma_\varrho}(t)$. By the definition of B we have

$$\overline{\overline{B \cap F_\varrho}} \leqq \overline{\overline{\bigcup_{\sigma<\sigma_\varrho} (B_\sigma \cap F_\varrho)}} + \overline{\overline{B_{\sigma_\varrho} \cap F_\varrho}} + \overline{\overline{\bigcup_{\sigma>\sigma_\varrho} (B_\sigma \cap F_\varrho)}}. \tag{x}$$

Taking into consideration that $B_\sigma \subseteq \tilde{S}_\sigma(t)$, we obtain from (5. 10. 3) that $\overline{\overline{\bigcup_{\sigma<\sigma_\varrho} (B_\sigma \cap F_\varrho)}} \leqq \overline{\overline{\bigcup_{\sigma<\sigma_\varrho} (\tilde{S}_\sigma(t) \cap F_\varrho)}} \leqq t$ and $\overline{\overline{\bigcup_{\sigma>\sigma_\varrho} (B_\sigma \cap F_\varrho)}} = 0$. On the other hand, it results from (5. 9) that $\tilde{S}_{\sigma_\varrho}(t) \cap F_\varrho = \tilde{F}_\varrho$ for every $F_\varrho \in \mathfrak{H}_{\sigma_\varrho}(t)$, hence $\overline{\overline{B \cap F_\varrho}} = \overline{\overline{B_{\sigma_\varrho} \cap \tilde{F}_\varrho}} < s$. It follows that $\overline{\overline{B \cap F_\varrho}} < t^+ + s$ for every $\varrho < \omega_\beta$. (1) and (2) mean that $\mathfrak{F}$ possesses property $\mathbf{B}(t^+ + s)$. Suppose now that t is finite. The formula (x) holds in this case too. We get from (5. 10. 3) that the first cardinal number on the right-hand side is less than t and the third one is 0, while the second is by the assumption less than s in this case too. Now if s is infinite, then the sum is less than s, hence less than $(t-1)+s$. If s is finite, then the first summand being less than t is at most $t-1$, hence the sum is less than $(t-1)+s$ in this case too. It results from (1) that if t is finite, then $\mathfrak{F}$ possesses property $\mathbf{B}((t-1)+s)$.

LEMMA 6. *The family $\mathfrak{F}$ possesses property $\mathbf{B}$, provided the same holds for the families $\tilde{\mathfrak{H}}_\sigma(t)$ for every $\sigma<\varphi$.*

PROOF. Lemma 6 is to be seen quite similarly to Lemma 5. Let $B_\sigma \subseteq \tilde{S}_\sigma(t)$ denote the sets satisfying the condition $B_\sigma \cap \tilde{F}_\varrho \neq 0$, $\tilde{F}_\varrho \subseteq\!\!\!\!\!\not\,\,\,\, B_\sigma$ for every $\tilde{F}_\varrho \in \tilde{\mathfrak{H}}_\sigma(t)$. Put $B=\bigcup_{\sigma<\varphi} B_\sigma$. The proof of the fact that B intersects every F_ϱ is the same as in Lemma 5. Let σ_ϱ denote, as before, the uniquely determined σ for which $\tilde{F}_\varrho \in \tilde{\mathfrak{H}}_{\sigma_\varrho}(t)$. It results from the definition (5. 9) and from (5. 10. 3) that $B \cap \tilde{F}_\varrho = B_{\sigma_\varrho} \cap \tilde{F}_\varrho$, hence $\tilde{F}_\varrho \neq B \cap \tilde{F}_\varrho$, since $\tilde{F}_\varrho \not\subseteq B_{\sigma_\varrho}$, and thus $\tilde{F}_\varrho \not\subseteq B$, therefore $F_\varrho \not\subseteq B$ for every $\varrho<\varphi$.

For the sake of brevity we introduce the following notations:

DEF. (5. 11) The cardinal number m is said to possess property $\mathbf{T}(p, r)$ if there exists an m' $(p \leqq m' < m)$ such that m' satisfies the formula (vv) of Lemma 4, i. e. if there exist ordinal numbers β' and γ such that

$$m'=\aleph_{\beta'}, \quad r=\aleph_\gamma \quad \text{and} \quad cf(\beta')=cf(\gamma).$$

Quite similarly, p is said to possess property $\mathbf{Q}(r)$ if p satisfies the formula (v) of (4. 9), i. e. if there exist ordinal numbers α_1 and γ such that

$$p=\aleph_\alpha=\aleph_{\alpha_1+1}, \quad r=\aleph_\gamma, \quad cf(\alpha_1)=cf(\gamma) \quad \text{and} \quad \gamma<\alpha_1 .$$

Now we are going to prove

(*) LEMMA 7. *$p(\overline{\tilde{\mathfrak{H}}_\sigma(t)})=p$, the families $\mathfrak{H}_\sigma(t)$ possess property $\mathbf{C}(q, r)$ and $\overline{\overline{\tilde{\mathfrak{H}}_\sigma(t)}}<m$ for every $\sigma<\varphi$, provided one of the conditions (δ) and $(\delta\delta)$ holds:*

(δ) $r=t$ and m does not possess property $\mathbf{T}(p, r)$.

$(\delta\delta)$ $r<t$.

(If t is finite, the hypothesis (*) is not used.)

PROOF. The first two statements were proved in (5. 10. 1) and (5. 10. 2). We have to prove the third one. It is obvious from the definitions (5. 7) and (5. 9) that $\overline{\overline{\tilde{\mathfrak{H}}_\sigma(t)}} \leqq \overline{\overline{\mathfrak{H}_\sigma(t)}} \leqq \overline{\overline{\mathfrak{F}_\sigma(t)}}$. We prove by induction on σ that $\overline{\overline{\mathfrak{F}_\sigma(t)}} \leqq \leqq p \cdot \overline{\sigma+1}<m$ for every $\sigma<\varphi$.

By the definition (5. 5) $\overline{\overline{\mathfrak{F}_0'(t)}}=\aleph_\alpha=p$ and, since by the assumption either $r<t$ or m possesses property $\mathbf{T}(p, r)$, by Lemma 4 $\overline{\overline{\mathfrak{F}_0(t)}}= =\overline{\overline{\mathrm{Clos}\,(\mathfrak{F}_0'(t), t)}}=p$.

Suppose that we have $\overline{\overline{\mathfrak{F}_{\sigma'}(t)}} \leqq p \cdot \overline{\sigma'+1}$ for every $\sigma' < \sigma$ for a $0 < \sigma < \varphi$. Then by (5.5)

$$\overline{\overline{\mathfrak{F}_\sigma^*}} \leqq \sum_{\sigma' < \sigma} \overline{\overline{\mathfrak{F}_{\sigma'}(t)}} \leqq \sum_{\sigma' < \sigma} p \cdot \overline{\sigma'+1} = p \cdot \overline{\sigma}.$$

Now $\overline{\overline{\mathfrak{F}_\sigma'(t)}} = \overline{\overline{\mathfrak{F}_\sigma^*(t)}} + 1 \leqq p \cdot \overline{\sigma+1}$.

We have $\varphi \leqq \omega_\beta$ from the definition (5.5), and therefore $p \cdot \overline{\sigma+1} < m$, hence we may apply Lemma 4 again to $\mathfrak{F}_\sigma(t) = \mathrm{Clos}\,(\mathfrak{F}_\sigma'(t), t)$ and we obtain $\overline{\overline{\mathfrak{F}_\sigma(t)}} \leqq p \cdot \overline{\sigma+1}$, thus this statement is proved for every $\sigma < \varphi$ and Lemma 7 is proved.

Note that from the statement $\overline{\overline{\mathfrak{F}_\sigma(t)}} \leqq p \cdot \overline{\sigma+1}$ $(\sigma < \varphi)$ it results that $\varphi = \omega_\beta$, but we do not use this fact.

Finally, to have a view of our results we need the following quite evident

LEMMA 8. *The least cardinal number which possesses property* $\mathbf{T}(\aleph_\alpha, \aleph_\gamma)$ $(\alpha > \gamma)$ *is* $\alpha + 1$ *if* $cf(\alpha) = cf(\gamma)$, *and it is* $\aleph_{\alpha + \omega_{cf(\gamma)} + 1}$ *if* $cf(\alpha) \neq cf(\gamma)$.

PROOF. By the definition (5.11) we have to find the least β_1 for which there exists a β' such that $\alpha \leqq \beta' < \beta_1$ and $cf(\beta') = cf(\gamma)$. It is obvious that $\beta_1 = \beta' + 1$ for the least ordinal number β' satisfying this condition, and $\beta' = \alpha$ if $cf(\alpha) = cf(\gamma)$.

Suppose now $cf(\alpha) \neq cf(\gamma)$. $\beta' > \alpha$ has the form $\beta' = \alpha + \beta''$ and $cf(\alpha + \beta'') = cf(\gamma)$ can hold only if β'' is of the second kind. But then $cf(\alpha + \beta'') = cf(\beta'')$ and the least ordinal number β'' of the second kind satisfying $cf(\beta'') = cf(\gamma)$ is $\omega_{cf(\gamma)}$.

Let for the sake of brevity $\tau(\alpha, \gamma)$ denote the index of the least cardinal number which possesses property $\mathbf{T}(\aleph_\alpha, \aleph_\gamma)$.

EXAMPLES.

$$\tau(n, 0) = \omega + 1, \quad \tau(\omega, 0) = \omega + 1, \quad \tau(\omega + 1, 0) = \omega \cdot 2 + 1;$$

or more generally

$$\tau(\alpha + \mu, \gamma) = \alpha + \omega_\gamma + 1 \quad \text{for} \quad 1 \leqq \mu \leqq \omega_\gamma \quad \text{if} \quad \gamma \leqq \alpha \quad \text{and} \quad \omega_\gamma \quad \text{is regular.}$$

6. Proof of the results concerning the conjectures (o) and (oo).

(*) THEOREM 4. *Suppose* $p \geqq \aleph_0$, $2 \leqq q \leqq p^+$ *and* $r^+ < p$. *Then for every cardinal number* m.

$$\mathbf{M}(m, p, q, r) \to \mathbf{B}.$$

(Note that if r is finite, the hypothesis (*) is not used.)

PROOF. For $m \leqq p$ the theorem follows from Theorem 2 (BERNSTEIN's theorem) if we use that symbol-I is decreasing in m (by (3.1)). We prove it by induction on m for every $m>p$. Suppose that the theorem is true for every $m'<m$. Let now $\mathfrak{F}$ be a family $(p(\mathfrak{F})=p,\ \overline{\overline{\mathfrak{F}}}=m)$ which possesses property $\mathbf{C}(q,r)$.

Put $t=r^+$. Then the conditions (°) are satisfied for the cardinal numbers in question and $r<t$. Hence we can carry out the construction described in Section 5 and we can apply Lemma 7. It results that the families $\widetilde{\mathcal{H}}_\sigma(t)$ possess property $\mathbf{C}(q,r)$, $p(\widetilde{\mathcal{H}}_\sigma(t))=p$ and $\overline{\overline{\widetilde{\mathcal{H}}_\sigma(t)}}<m$ for every $\sigma<\varphi$. Using the induction hypothesis we obtain that the families $\widetilde{\mathcal{H}}_\sigma(t)$ possess property $\mathbf{B}$ and thus by Lemma 6 the same holds for the family $\mathfrak{F}$ too. Q. e. d.

REMARK. Theorem 4 is clearly a generalization of Theorem 1 (MILLER's theorem) for infinite r's, however, it is not best-possible in r as we have already mentioned. It is possible that under the conditions $p\geqq\aleph_0$, $q\leqq p^+$ the theorem holds for every $r<p$. We have to deal only with the case $p=r^+$.

Here we can prove the following

(*) THEOREM 5. Suppose $r=\aleph_\gamma$, $r^+=p$ *(i. e.* $p=\aleph_\alpha=\aleph_{\gamma+1}$*)*, $2\leqq q\leqq p^+$. *Then* $\mathbf{M}(m,p,q,r)\rightarrow\mathbf{B}$ *holds for every* m *less than* $\aleph_{\gamma+\omega_{cf(\gamma)}+1}$.

PROOF. For $m\leqq p$ the theorem is true by Theorem 2. We prove it by induction on m for every $p<m<\aleph_{\gamma+\omega_{cf(\gamma)}+1}$. Suppose that it is true for every $m'<m$ for an m satisfying the above condition. Let $\mathfrak{F}$ be a family for which $p(\mathfrak{F})=p$, $\overline{\overline{\mathfrak{F}}}=m$ and suppose that $\mathfrak{F}$ possesses property $\mathbf{C}(q,r)$. Put $t=r$. The conditions (°) hold for the cardinal numbers in question, and so we can consider the families $\widetilde{\mathcal{H}}_\sigma(r)$ $(\sigma<\varphi)$ defined in (5.9). Since by the assumption $cf(\alpha)=cf(\gamma+1)$ $(cf(\alpha)\neq cf(\gamma))$, it follows from Lemma 8 that m does not possess property $\mathbf{T}(p,r)$. It results from Lemma 7 $(\delta\delta)$ that $p(\widetilde{\mathcal{H}}_\sigma(r))=p$, $\widetilde{\mathcal{H}}_\sigma(r)$ possesses property $\mathbf{C}(q,r)$ and $\overline{\overline{\widetilde{\mathcal{H}}_\sigma(r)}}<m$ for every $\sigma<\varphi$. Hence by the induction hypothesis the families $\widetilde{\mathcal{H}}_\sigma(r)$ possess property $\mathbf{B}$. Consequently, by Lemma 6, the same is true for $\mathfrak{F}$.

REMARK. We do not know for any γ whether the assumption $m<\aleph_{\gamma+\omega_{cf(\gamma)}+1}$ can be omitted. We have formulated the simplest unsolved problem in Section 3 (see Problem 2).

(*) THEOREM 6. *Suppose* $p>r\geqq\aleph_0$, *then* $\mathbf{M}(m,p,r)\rightarrow\mathbf{B}(r^{++})$ *for every* m.

PROOF. If $p=r^+$, then the theorem is trivially true by (3.2). Thus we may suppose $r^+<p$. In the cases $m<p$ by (4.2) we have $\mathbf{M}(m,p,r)\rightarrow\mathbf{B}(2)$.

If p does not possess property $\mathbf{Q}(r)$, then by (4.9) $\mathbf{M}(p,p,r)\rightarrow\mathbf{B}(r^+)$ holds. If p possesses property $\mathbf{Q}(r)$, then it obviously does not possess roperty $\mathbf{Q}(r^+)$ (since if $r=\aleph_\gamma$, then $r^+=\aleph_{\gamma+1}$ and $cf(\gamma)\neq cf(\gamma+1)$).

It follows again from (4.9) that $\mathbf{M}(p, p, r^+) \to \mathbf{B}(r^{++})$ holds. As a consequence of (3. 1) we get that $\mathbf{M}(p, p, r) \to \mathbf{B}(r^{++})$ holds in every cases. Now we prove the theorem for $m > p$ by induction on m as follows:

Suppose that it is true for every $m' < m$. Let $\mathfrak{F}$ be a family $(p(\mathfrak{F}) = p, \overline{\overline{\mathfrak{F}}} = m)$ which possesses property $\mathbf{C}(2, r)$. Put $t = r^+$. Then the conditions (°) hold for the cardinal numbers in question and we can consider the families $\widetilde{\mathfrak{H}}_\sigma(t)$ $(\sigma < \varphi)$. Since $r < t$, it results from Lemma 7 that $p(\widetilde{\mathfrak{H}}_\sigma(t)) = p$, the families $\widetilde{\mathfrak{H}}_\sigma(t)$ possess property $\mathbf{C}(2, r)$ and $\overline{\overline{\widetilde{\mathfrak{H}}_\sigma(t)}} < m$ for every $\sigma < \varphi$. Thus by the induction hypothesis the families $\widetilde{\mathfrak{H}}_\sigma(t)$ possess property $\mathbf{B}(r^{++})$. Applying Lemma 5 we obtain that $\mathfrak{F}$ possesses property $\mathbf{B}(r^{++} + r^{++})$, i. e. it possesses property $\mathbf{B}(r^{++})$.

REMARK. It is obvious from (3. 1) that under the conditions of Theorem 6 $\mathbf{M}(m, p, r) \to \mathbf{B}(s)$ holds for every $s \geqq r^{++}$ too. In the case $q = 2$ Theorem 4 is a corrollary of Theorem 6. Similarly as in the case of Theorem 4, it is possible that Theorem 6 holds with r^+ instead of r^{++}.

(*) THEOREM 7. *Suppose $p > r \geqq \aleph_0$. (Put $p = \aleph_\alpha$ $r = \aleph_\gamma$.) Suppose further that p does not possess property $\mathbf{Q}(r)$. Then*

$$\mathbf{M}(m, p, r) \to \mathbf{B}(r^+)$$

for every $m < \aleph_{\alpha+\omega_{cf(\gamma)}+1}$, provided $cf(\alpha) \neq cf(\gamma)$.

PROOF. For $m < p$ the theorem is a corollary of (4. 2). In the case $m = p$ we get from (4. 9) that $\mathbf{M}(p, p, r) \to \mathbf{B}(r^+)$ holds, since the assumption of our theorem assures that p and r do not satisfy the formula (v) of (4. 9).

We are going to prove our theorem for $m > p$ by induction on m as follows: Suppose that the theorem is true for every $m' < m$, for an m satisfying the above condition. Let $\mathfrak{F}$ be a family $(p(\mathfrak{F}) = p, \overline{\overline{\mathfrak{F}}} = m)$ which possesses property $\mathbf{C}(2, r)$. Put $t = r$. The conditions (°) are satisfied, and so we can consider the families $\widetilde{\mathfrak{H}}_\sigma(r)$. The assumption $cf(\alpha) \neq cf(\gamma)$ assures by Lemma 8 that m does not possess property $\mathbf{T}(p, r)$. Thus from Lemma 7 we obtain that $p(\widetilde{\mathfrak{H}}(r)) = p$, the families $\widetilde{\mathfrak{H}}_\sigma(r)$ possess property $\mathbf{C}(2, r)$ and $\overline{\overline{\widetilde{\mathfrak{H}}_\sigma(r)}} < m$ for every $\sigma < \varphi$.

Thus, by the induction hypothesis, the families $\widetilde{\mathfrak{H}}_\sigma(r)$ possess property $\mathbf{B}(r^+)$ and, consequently, by Lemma 5, the family $\mathfrak{F}$ possesses $\mathbf{B}(r^+ + r^+)$ Since r is supposed to be infinite, this means that $\mathfrak{F}$ possesses property $\mathbf{B}(r^+)$ too.

REMARKS. If p possesses property $\mathbf{Q}(r)$, we do not know whether the theorem is true for $m = p$. (See the remark after (4. 9) and Problem 3a).)

If $cf(\alpha)=cf(\gamma)$, then by (4.9) the theorem is true for $m=p$, but we do not know whether it is true for $m=p^+$ or not. The simplest unsolved problem here is $\mathbf{M}(\aleph_{\omega+1}, \aleph_\omega, \aleph_0)\to\mathbf{B}(\aleph_{\omega+1})$.

Here the difficulty is essentially the same as in Problem 3b). It is obvious from the remark made after (4.9) that a positive solution of Problem 3b) would imply the positive solution of the problem just stated as well as a positive solution of Problem 3a).

7. The discussion of symbol-II in the cases $r<\aleph_0$ $(p\geqq\aleph_0)$. Note that in the case $r<\aleph_0$ $(p\geqq\aleph_0)$ symbol-I is completely discussed by MILLER's theorem. The positive theorems concerning symbol-II will be proved by MILLER's method quite similarly as the theorems of Section 6.

THEOREM 8. a) $\mathbf{M}(\aleph_{\alpha+n}, \aleph_\alpha, r)\to\mathbf{B}((r-1)(n+1)+2)$ *if r is finite and α is arbitrary.*

b) $\mathbf{M}(m, \aleph_\alpha, r)\to\mathbf{B}(\aleph_0)$ *for every m and α, provided $r<\aleph_0$.*[10]

PROOF. a) We are going to prove the theorem by induction on n. For $n=0$ the theorem is proved in (4.9). Suppose that it is true for an n and let $\mathfrak{F}$ be a family such that $p(\mathfrak{F})=\aleph_\alpha$, $\overline{\overline{\mathfrak{F}}}=\aleph_{\alpha+n+1}$ and suppose that it possesses property $\mathbf{C}(2,r)$. It is obvious that the conditions (°) hold for the cardinal numbers in question and we can apply the construction of Section 5 with $t=r$ to our family $\mathfrak{F}$.

By Lemma 7, $p(\tilde{\mathfrak{H}}_\sigma(r))=p$, the families $\tilde{\mathfrak{H}}_\sigma(r)$ possess property $\mathbf{C}(2,r)$ and $\overline{\overline{\tilde{\mathfrak{H}}_\sigma(r)}}<\aleph_{\alpha+n+1}$ for every $\sigma<\varphi$. This means that $\overline{\overline{\tilde{\mathfrak{H}}_\sigma(r)}}\leqq\aleph_{\alpha+n}$ for every $\sigma<\varphi$ and — using (3.1) — we get from the induction hypothesis that the families $\tilde{\mathfrak{H}}_\sigma(r)$ possess property $\mathbf{B}((r-1)(n+1)+2)$ for every $\sigma<\varphi$.

It follows from Lemma 5 that the family $\mathfrak{F}$ possesses property $\mathbf{B}((r-1)+ +(r-1)(n+1)+2)$, i. e. it possesses property $\mathbf{B}((r-1)(n+2)+2)$.

b) The proof can be carried out by induction on m using Lemmas 5 and 7 quite similarly as in the previous cases, and so we omit the proof.

REMARK. The hypothesis (∗) is not used in the proof, since it is not used in the proof of Lemma 7 for the case of finite r.

With a slight modification of our construction it would be easy to prove the following

THEOREM 9. *Let $\mathfrak{F}$ be a family, $p(\mathfrak{F})=\aleph_\alpha$, $\overline{\overline{\mathfrak{F}}}=\aleph_{\alpha+n}$, and suppose that it possesses property $\mathbf{C}(2,r)$ for a finite r where α is arbitrary.*

Let there be given a function $l(F)$ which correlates to every $F\in\mathfrak{F}$ an integer $l(F)$.

[10] Note that n denotes always a non-negative integer and r is supposed to be greater than 0.

Then there exists a set B such that

$$\overline{\overline{B \cap F}} = \max\bigl(l(F), (r-1)(n+1)+1\bigr) \quad \textit{for every} \quad F \in \mathfrak{F}.$$

In particular, if $l(F) \equiv (r-1)(n+1)+1$, then the set B intersects every F in exactly $(r-1)(n+1)+1$ points.

We omit the proof.

Now we are going to prove that Theorem 8 is best-possible in s.

(*) THEOREM 10. a) $\mathbf{M}(\aleph_{\alpha+n}, \aleph_\alpha, r) \not\to \mathbf{B}\bigl((r-1)(n+1)+1\bigr)$ *if r is finite and α is arbitrary.*

b) $\mathbf{M}(m, \aleph_\alpha, r) \not\to \mathbf{B}(l)$ *if $r > 1$ is finite, α is arbitrary, $m \geqq \aleph_{\alpha+\omega}$ and l is an integer.*

PROOF. a) We have to prove that there exists a family $\mathfrak{F}$ satisfying the following conditions:

(1) $p(\mathfrak{F}) = \aleph_\alpha$.

(2) $\overline{\overline{\mathfrak{F}}} = \aleph_{\alpha+n}$.

(3) $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$.

(4) If for a set B $B \cap F \neq 0$ for every $F \in \mathfrak{F}$, then there exists an $F_0 \in \mathfrak{F}$ such that $\overline{\overline{F_0 \cap B}} \geqq (r-1)(n+1)+1$.

We are going to prove instead of this the following more general statement: There exists a family $\mathfrak{F}$ satisfying the conditions (1), (2), (3) and the following condition:

(5) There exist subfamilies $\mathfrak{F}_1$, $\mathfrak{F}_2$ of $\mathfrak{F}$ such that $\mathfrak{F}_1 \cup \mathfrak{F}_2 = \mathfrak{F}$, $\mathfrak{F}_1 \cap \mathfrak{F}_2 = 0$ and if for a set B $B \cap F \neq 0$ for every $F \in \mathfrak{F}_1$, then there exists an $F_0 \in \mathfrak{F}_2$ such that $\overline{\overline{F_0 \cap B}} \geqq (r-1)(n+1)+1$.

It is obvious that (5) implies (4).

Put $(\mathfrak{F}) = S$. Obviously (1) and (2) imply $\overline{\overline{S}} \leqq \aleph_{\alpha+n}$. Thus we have:

(6) If there exists a family $\mathfrak{F}$ satisfying the conditions (1), (2), (3) and (5), then for an arbitrary set S' ($\overline{\overline{S'}} = \aleph_{\alpha+n}$) there exists a family $\mathfrak{F}'$ such that $(\mathfrak{F}') \subseteq S'$ and $\mathfrak{F}'$ satisfies the conditions (1), (2), (3) and (5) too.

We prove the existence of such a family $\mathfrak{F}$ by induction on n. For $n = 0$ the theorem is proved in (4.5) (see the remark after (4.5)).[11] Suppose that for an n there exists a family $\mathfrak{F}$ satisfying the formulas (1), (2), (3) and (5). Let S be a set, $\overline{\overline{S}} = \aleph_{\alpha+n+1}$. Then $\overline{\overline{[S]^{\aleph_{\alpha+n}}}} = \aleph_{\alpha+n+1}$ by the hypothesis (*). Let $\{A_\varrho\}_{\varrho < \omega_{\alpha+n+1}} = [S]^{\aleph_{\alpha+n}}$ be a well-ordering of type $\omega_{\alpha+n+1}$ of the set $[S]^{\aleph_{\alpha+n}}$.

[11] In case of finite r the construction given in (4.5) can be simplified as follows: Suppose that $\overline{\overline{\mathfrak{F}_1}} = r$ instead of $\overline{\overline{\mathfrak{F}_1}} = r^+$ and take for $\mathfrak{S}$ the system of all subsets X of $(\mathfrak{F}_1)$ satisfying the condition $\overline{\overline{X \cap F}} = 1$ for every $F \in \mathfrak{F}_1$ instead of the system $\mathfrak{S}$ defined in (4.5.4).

We are going to define a sequence $\{\mathfrak{F}_\varrho\}_{\varrho<\omega_{\alpha+n+1}}$ of type $\omega_{\alpha+n+1}$ of families $(\mathfrak{F}_\varrho)\subseteq S$ by induction on ϱ as follows:

Suppose that the families $\mathfrak{F}_{\varrho'}$ are defined for a $\varrho<\omega_{\alpha+n+1}$ in such a way that $\overline{\overline{(\mathfrak{F}_{\varrho'})}}\leqq\aleph_{\alpha+n}$ for every $\varrho'<\varrho$. Then $\overline{\overline{A_\varrho\cup\bigcup_{\varrho'<\varrho}(\mathfrak{F}_{\varrho'})}}\leqq\aleph_{\alpha+n}$, hence we can define a subset S_ϱ of S such that

$$S_\varrho\subseteq S-\big(A_\varrho\cup\bigcup_{\varrho'<\varrho}(\mathfrak{F}_{\varrho'})\big) \text{ and } \overline{\overline{S}}_\varrho=\aleph_{\alpha+n}. \tag{7}$$

By the induction hypothesis and by (6) there exists a family $\mathfrak{F}_\varrho^*$ satisfying the formulas (1), (2), (3) and (5) such that

$$(\mathfrak{F}_\varrho^*)\subseteq S_\varrho; \tag{8}$$

let $\mathfrak{F}_\varrho^{1,*}$ and $\mathfrak{F}_\varrho^{2,*}$ denote the families satisfying (5) instead of $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively.

Since $\mathfrak{F}_\varrho^*$ satisfies (2), we have $\overline{\overline{\mathfrak{F}_\varrho^{2,*}}}\leqq\aleph_{\alpha+n}$ and we may suppose that the equality holds. Let $\mathfrak{F}_\varrho^{2,*}=\{F_\nu^{\varrho,2,*}\}_{\nu<\omega_{\alpha+n}}$ be a well-ordering of type $\omega_{\alpha+n}$ of the family $\mathfrak{F}_\varrho^{2,*}$.

Since $\overline{\overline{A}}_\varrho=\aleph_{\alpha+n}$, it is obvious that there exists a system $\mathbb{S}_\varrho$ of subsets of A_ϱ satisfying the following conditions:

(9) $\overline{\overline{\mathbb{S}}}_\varrho=\aleph_{\alpha+n}$, $\overline{\overline{X}}=r-1$ for every $X\in\mathbb{S}$ and $X\cap Y=0$ for every $X, Y\in\mathbb{S}, X\neq Y$.

Let $\mathbb{S}_\varrho=\{X_\nu^\varrho\}_{\nu<\omega_{\alpha+n}}$ be a well-ordering of type $\omega_{\alpha+n}$ of the set $\mathbb{S}_\varrho$.

We define the families $\mathfrak{F}_\varrho$, $\mathfrak{F}_\varrho^1$, $\mathfrak{F}_\varrho^2$ by the following formulas:

$$\mathfrak{F}_\varrho^1=\mathfrak{F}_\varrho^{1,*},\quad \mathfrak{F}_\varrho^2=\{X_\nu^\varrho\cup F_\nu^{\varrho,2,*}\}_{\nu<\omega_{\alpha+n}} \tag{10}$$

and

$$\mathfrak{F}_\varrho=\mathfrak{F}_\varrho^1\cup\mathfrak{F}_\varrho^2.$$

It is obvious that $(\mathfrak{F}_\varrho)\subseteq S_\varrho+A_\varrho$, hence $\overline{\overline{(\mathfrak{F}_\varrho)}}\leqq\aleph_{\alpha+n}$, and so $\mathfrak{F}_\varrho$ is defined for every $\varrho<\omega_{\alpha+n+1}$ and the formulas (7)—(10) are satisfied for every $\varrho<\omega_{\alpha+\nu+1}$.

Put

$$\mathfrak{F}=\bigcup_{\varrho<\omega_{\alpha+n+1}}\mathfrak{F}_\varrho,\quad \mathfrak{F}^1=\bigcup_{\varrho<\omega_{\alpha+n+1}}\mathfrak{F}_\varrho^1,\quad \mathfrak{F}^2=\bigcup_{\varrho<\omega_{\alpha+n+1}}\mathfrak{F}_\varrho^2. \tag{11}$$

Now we have to verify that $\mathfrak{F}$ satisfies (1), (2), (3) and (5) for $n+1$ instead of n.

$p(\mathfrak{F}_\varrho^*)=\aleph_\alpha$, since $\mathfrak{F}_\varrho^*$ satisfies (1), and thus it follows immediately from the definitions (9), (10) and (11) that

$$p(\mathfrak{F})=\aleph_\alpha. \tag{12}$$

It results immediately from (7) and (8) that

(13) $$(\mathfrak{F}^1_{\varrho'}) \cap (\mathfrak{F}^1_{\varrho}) = 0 \quad \text{if} \quad \varrho' < \varrho < \omega_{\alpha+n+1}.$$

Thus, since the families $\mathfrak{F}^1_\varrho$ are non-empty, we have

(14) $$\overline{\overline{\mathfrak{F}}} = \aleph_{\alpha+n-1}.$$

Now we prove:

(15) $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$.

Let F, F' be two distinct elements of $\mathfrak{F}$. Then $F \in \mathfrak{F}_\varrho$ and $F' \in \mathfrak{F}_{\varrho'}$ for suitable ϱ and ϱ', respectively. We distinguish two cases: (i) $\varrho = \varrho'$, (ii) $\varrho \neq \varrho'$.

(i) If $F \in \mathfrak{F}^1_\varrho$, $F' \in \mathfrak{F}^1_\varrho$, then $\overline{\overline{F \cup F'}} < r$, since by (10) $\mathfrak{F}^1_\varrho = \mathfrak{F}^{1,*}_\varrho$ and $\mathfrak{F}^*_\varrho$ satisfies (3).

If $F \in \mathfrak{F}^1_\varrho$, $F' \in \mathfrak{F}^2_\varrho$, then $F \subseteq S_\varrho$, $F' = X^\varrho_\nu \cup F^{\varrho,2,*}_\nu$ for a suitable $\nu < \omega_{\alpha+n}$, but by (9) $X^\varrho_\nu \subseteq A_\varrho$, hence by (7) and (8) $F \cap F' = F \cap F^{\varrho,2,*}_\nu$ and $\overline{\overline{F \cap F'}} < r$ follows again from the fact that $\mathfrak{F}^*_\varrho$ satisfies (3).

If $F \in \mathfrak{F}^2_\varrho$, $F' \in \mathfrak{F}^2_\varrho$, then $F = X^\varrho_\nu \cup F^{\varrho,2,*}_\nu$, $F' = X^\varrho_{\nu'} \cup F^{\varrho,2,*}_{\nu'}$ for suitable ν', ν $(\nu \neq \nu')$, respectively. Using again that A_ϱ and S_ϱ are disjoint, we get

$$F \cap F' = (X^\varrho_\nu \cap X^\varrho_{\nu'}) \cup (F^{\varrho,2,*}_\nu \cap F^{\varrho,2,*}_{\nu'}).$$

Thus, using that by (9) $X^\varrho_\nu \cap X^\varrho_{\nu'} = 0$, we get by the same argument as above that $\overline{\overline{F \cap F'}} < r$ in this case too.

(ii) We may suppose $\varrho' < \varrho$. If $F \in \mathfrak{F}^1_\varrho$, then by (7), (8) and (10) F and F' are disjoint. If $F \in \mathfrak{F}^2_\varrho$, then $F = X^\varrho_\nu \cup F^{\varrho,2,*}_\nu$ for a suitable ν and it results from (7) and (8) that $F' \cap F \subseteq X^\varrho_\nu$, hence by (9) $\overline{\overline{F' \cap F}} \leqq r-1 < r$.

We have

(16) $$\mathfrak{F}^1 \cap \mathfrak{F}^2 = 0.$$

In fact, $\mathfrak{F}^{1,*}_\varrho \cap \mathfrak{F}^{2,*}_\varrho = 0$ for every ϱ, because $\mathfrak{F}^*_\varrho$ satisfies (5), thus it results from the definition (10) and e. g. from the fact that $\mathfrak{F}^*_\varrho$ satisfies (3), that $\mathfrak{F}^1_\varrho \cap \mathfrak{F}^2_\varrho = 0$ and it is obvious from (7) and (8) that $\mathfrak{F}_{\varrho'} \cap \mathfrak{F}_\varrho = 0$ for $\varrho' \neq \varrho$, hence $\mathfrak{F}_1 \cap \mathfrak{F}_2 = 0$ is true.

Now we prove:

(17) Suppose that for a set B $B \cap F \neq 0$ for every $F \in \mathfrak{F}^1$.

Then there exists an $F_0 \in \mathfrak{F}_2$ such that $\overline{\overline{F_0 \cap B}} \geqq (r-1)(n+2)+1$.

First of all it follows from (13) that $\overline{\overline{B}} = \aleph_{\alpha+n+1}$. As a corollary of this there exists a subscript ϱ_0 such that $A_{\varrho_0} \subseteq B$. Since by the assumption B has to intersect every $F \in \mathfrak{F}^1$, we have that $F \cap B \neq 0$ for every $F \in \mathfrak{F}^1_{\varrho_0}$. But by (10) $\mathfrak{F}^1_\varrho = \mathfrak{F}^{1,*}_\varrho$ and it follows from the fact that $\mathfrak{F}^*_{\varrho_0}$, $\mathfrak{F}^{1,*}_{\varrho_0}$, $\mathfrak{F}^{2,*}_{\varrho_0}$ satisfy (5), that there exists an index ν_0 such that $\overline{\overline{B \cap F^{\varrho_0,2,*}_{\nu_0}}} \geqq (r-1)(n+1)+1$. Put

$F^0 = X_{\nu_0}^{\varrho_0} \cup F_{\nu_0}^{\varrho_0, 2, *}$. Then $F^0 \in \mathfrak{F}^2$. Taking into consideration that by (7) and (8) $X_{\nu_0}^{\varrho_0} \cap F_{\nu_0}^{\varrho_0, 2, *} = 0$ and by (9) $X_{\nu_0}^{\varrho_0} \subseteq A_{\varrho_0} \subseteq B$, $\overline{\overline{X_{\nu_0}^{\varrho}}} = r - 1$, we obtain

$$\overline{\overline{B \cap F^0}} \geqq \overline{\overline{X_{\nu_0}^{\varrho_0}}} + \overline{\overline{B \cap F_{\nu_0}^{\varrho_0, 2, *}}} \geqq (r-1)(n+2)+1.$$

Thus the families $\mathfrak{F}$, $\mathfrak{F}^1$ and $\mathfrak{F}^2$ satisfy by (12), (14), (15), (16) and (17) the formulas (1), (2), (3) and (5) for $n+1$ instead of n, and so the existence of such a family is proved for every n.

b) By (3.1) it suffices to prove that $\mathbf{M}(\aleph_{\alpha+\omega}, \aleph_\alpha, r) \not\rightarrow \mathbf{B}(l)$.

Let $\{S_n\}_{n<\omega}$ be a sequence of disjoint sets such that $\overline{\overline{S_n}} = \aleph_{\alpha+n}$. By the theorem just proved and by the remark (6) there exists a sequence $\{\mathfrak{F}_n\}_{n<\omega}$ of families such that $(\mathfrak{F}_n) \subseteq S_n$ and $\mathfrak{F}_n$ satisfies for every n the conditions (1), (2), (3) and (5).

Put $\mathfrak{F} = \bigcup\limits_{n<\omega} \mathfrak{F}_n$. Then $p(\mathfrak{F}) = \aleph_\alpha$ and $\overline{\overline{\mathfrak{F}}} = \aleph_{\alpha+\omega}$, since the $\mathfrak{F}_n$'s satisfy (1) and (2) for every n and the $\mathfrak{F}_n$'s are obviously disjoint.

Since the sets S_n are disjoint, $F \cap F' = 0$, provided $F \in \mathfrak{F}_n$, $F' \in \mathfrak{F}_{n'}$ for $n \neq n'$. Thus, taking into consideration that $\mathfrak{F}_n$ satisfies (3) for every n, it follows that $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$. But $\mathfrak{F}$ does not possess property $\mathbf{B}(l)$ for any l, since there exists an n_0 such that $(r-1)(n_0+1)+1 > l$ and the subfamily $\mathfrak{F}_{n_0}$ of $\mathfrak{F}$ does not possess property $\mathbf{B}((n-1)(n_0+1)+1)$, because it satisfies (5).

Thus part b) of Theorem 10 is also proved.

REMARK. As we have already mentioned in (4.5), in the case $n=0$ of the part a) of Theorem 10 the hypothesis (*) is not used. We do not even know whether one can prove Theorem 10a) for $n=1$ without using (*).

8. Results on the topological products. A topological space $\mathfrak{X}$ is said to be $\varkappa$-compact if every family $\mathfrak{M}$ of closed subsets of it with void intersection, $\bigcap\limits_{X \in \mathfrak{M}} X = 0$, contains a subfamily $\mathfrak{M}' \subseteq \mathfrak{M}$ $(\overline{\overline{\mathfrak{M}'}} < \aleph_\varkappa)$ with void intersection.

0-compactness means ordinary compactness.

1-compact spaces are the Lindelöf spaces.

For the sake of brevity we introduce the symbol $\boldsymbol{T}(m, \lambda) \rightarrow \varkappa$ to indicate the following statement:

If $\mathfrak{F}$ is a family of λ-compact *discrete* topological spaces, $\overline{\overline{\mathfrak{F}}} = m$, then the topological product of the elements of $\mathfrak{F}$ is $\varkappa$-compact.

As usual, $\boldsymbol{T}(m, \lambda) \not\rightarrow \varkappa$ denotes the negation of this statement.

TYCHONOV's classical theorem can be stated as follows: $\boldsymbol{T}(m, 0) \rightarrow 0$ for every cardinal number m.

Let S be a set, $\overline{\overline{S}} = m$, and let $\mu(x)$ be a measure defined on all subsets of S such that the values of $\mu(x)$ are 0 and 1, $\mu(\{x\}) = 0$ for every $x \in S$.

The cardinal number m is said to be of measure 0 if every σ-measure satisfying the above condition vanishes identically.[12]

A well-known result of ULAM states that every cardinal number m less than the first strongly inaccessible aleph is of measure 0.[13]

The hypothesis (**) states that a strongly inaccessible $> \aleph_0$ aleph is not of measure 0 or more generally:

(**) If m is strongly inaccessible, $> \aleph_0$, then there exists an m-additive measure satisfying the above conditions such that $\mu(S) = 1$.

If we use (*), then ŁOS's theorem (Theorem 4 of [3]) states that

$$T(\aleph_{\varkappa+1}, 1) \not\to \varkappa \quad \text{for every} \quad \varkappa \geqq 1,$$

provided $\aleph_\varkappa$ is regular and of measure 0.[14]

Now we are going to prove the following

(*) THEOREM 11. *$T(\aleph_{\alpha+n}, \alpha+1) \not\to \alpha+n$ for every ordinal number α and for every $1 \leqq n < \omega$.*

Before proving this theorem[15] we compare it with ŁOS's theorem and state the simplest unsolved problems. Put $\alpha = 0$, then our theorem states that $T(\aleph_n, 1) \not\to n$ for every $n \geqq 1$, and so it is stronger than ŁOS's theorem for the cases $\varkappa < \omega$. Moreover it is best-possible, namely $T(\aleph_n, 1) \to n+1$ is trivially true, since the topological product of $\aleph_\alpha$ Lindelöf spaces contains a base of power $\aleph_\alpha$ for every α. For the case of singular $\varkappa$'s, e. g. for $\varkappa = \omega$ the following problem remains open:

PROBLEM 4. $T(\aleph_\omega, 1) \to \omega$?

($T(\aleph_{\omega, 1}) \to \omega + 1$ is trivially true and $T(\aleph_\omega, 1) \not\to n$ for every finite n is a consequence of both theorems.)

For $\varkappa$'s greater than ω ŁOS's theorem is stronger, since our result states nothing about $\varkappa$-compactness of the product of Lindelöf spaces for $\varkappa > \omega$.

But we do not know whether ŁOS's theorem is best-possible e. g. for $\varkappa = \omega + 1$, since it states $T(\aleph_{\omega+2}, 1) \to \omega + 1$ and the following problem remains open:

PROBLEM 5. $T(\aleph_{\omega+2}, 1) \to \omega + 2$?

(Our Theorem 11 gives only that $T(\aleph_{\omega+2}, \omega+1) \not\to \omega + 2$.)

[12] See [3], p. 14.

[13] See [7].

[14] See [3], Theorem 4, p. 17.

[15] The proof is given on p. 115.

For $\aleph_\varkappa$'s not less than the first inaccessible cardinal number Łoś's theorem does not state anything. The reason for this is that if, at least, we assume the hypothesis (**), then $T(m_0, 1) \to \alpha_0$ is true where $m_0 = \aleph_{\alpha_0}$ denotes the first strongly inaccessible cardinal number $> \aleph_0$. More generally we have the following

(**) THEOREM 12. *If* $\aleph_\alpha$ *is strongly inaccessible*, $> \aleph_0$, *then*

$$T(\aleph_\alpha, \alpha) \to \alpha.$$ [16]

We mention here that even using (*) and (**) we can not decide whether $T(\aleph_\alpha, 1) \to \alpha_0$ is true if $\alpha > \alpha_0$ where $\aleph_{\alpha_0}$ is the first inaccessible cardinal number $> \aleph_0$.

Our theorem shows that $T(\aleph_{\alpha_0} + n, \alpha_0 + 1) \not\to \alpha_0 + n$ for every $1 \leqq n < \omega$, but neither Łoś's theorem nor our theorem disproves that $T(m, \alpha_0 + 1) \to \alpha_0 + \omega$ holds for every cardinal number m if $\aleph_{\alpha_0}$ is strongly inaccessible $> \aleph_0$.

PROOF OF THEOREM 11. Let r_0 be an integer such that $(r_0 - 1)(n+1) + 1 \geqq (r_0 - 1)n + 2$ (e. g. $r_0 = 2$). By Theorem 10 corresponding to every n there exists a family $\mathfrak{F}$ $((\mathfrak{F}) = S)$ satisfying the following conditions:

(1) $p(\mathfrak{F}) = \aleph_\alpha$.

(2) $\overline{\overline{\mathfrak{F}}} = \aleph_{\alpha+n}$.

(3) $\mathfrak{F}$ possesses property $\mathbf{C}(2, r_0)$.

(4) If for a set B $B \cap F \neq 0$ for every $F \in \mathfrak{F}$, then there exists an $F_0 \in \mathfrak{F}$, such that

$$\overline{\overline{F_0 \cap B}} \geqq (r_0 - 1)(n+1) + 1.$$

Let $\mathfrak{F} = \{F_\varrho\}_{\varrho < \omega_{\alpha+n}}$ be a well-ordering of type $\omega_{\alpha+n}$ of $\mathfrak{F}$. Let $\mathfrak{X}$ denote the topological product of the discrete spaces F_ϱ. The elements of $\mathfrak{X}$ are the sequences $(x_\varrho)_{\varrho < \omega_{\alpha+n}}$ where $x_\varrho \in F_\varrho$.

Corresponding to every finite sequence $\varrho_1 < \cdots < \varrho_k < \omega_{\alpha+n}$ we define the subset $B_{\varrho_1 \ldots \varrho_k}((x_\varrho)_{\varrho < \omega_{\alpha+n}})$ of S as the set of ϱ_ith components of $(x_\varrho)_{\varrho < \omega_{\alpha+n}}$ for $i = 1, \ldots, k$, i. e. we put

$$B_{\varrho_1 \ldots \varrho_k}((x_\varrho)_{\varrho < \omega_{\alpha+n}}) = \{x : x \in S \wedge (x = x_{\varrho_1} \vee \cdots \vee x = x_{\varrho_k})\}. \tag{5}$$

Now we define the subset $X_{\varrho_1 \ldots \varrho_k}$ of $\mathfrak{X}$ as follows:

(6) $$(x_\varrho)_{\varrho < \omega_{\alpha+n}} \in X_{\varrho_1 \ldots \varrho_k} \quad \text{if and only if}$$

$$\overline{\overline{B_{\varrho_1 \ldots \varrho_k}((x_\varrho)_{\varrho < \omega_{\alpha+n}}) \cap F_{\varrho_i}}} < (r_0 - 1)n + 2 \quad \text{for every} \quad i = 1, \ldots, k.$$

[16] For the proof see p. 116.

Put

$$\mathfrak{M} = \{X_{\varrho_1 \dots \varrho_k}\}_{(\varrho_1 < \dots < \varrho_k < \omega_{\alpha+n})}.$$

It is obvious that $X_{\varrho_1 \dots \varrho_k}$ is a closed subset of $\mathfrak{X}$ for every sequence $\varrho_1 < \dots < \varrho_k < \omega_{\alpha+n}$ and it results from (1) that the discrete spaces F_ϱ are $\aleph_{\alpha+1}$-compact for every $\varrho < \omega_{\alpha+n}$. Hence it is enough to prove the following assertions:

(7)
$$\bigcap_{X_{\varrho_1 \dots \varrho_k} \in \mathfrak{M}} X_{\varrho_1 \dots \varrho_k} = 0$$

and

(8)
$$\bigcap_{X_{\varrho_1 \dots \varrho_k} \in \mathfrak{M}'} X_{\varrho_1 \dots \varrho_k} \neq 0 \quad \text{if} \quad \mathfrak{M}' \subseteq \mathfrak{M}, \quad \overline{\overline{\mathfrak{M}'}} < \aleph_{\alpha+n}.$$

Proof of (7). Let $(x_\varrho^0)_{\varrho < \omega_{\alpha+n}}$ be an arbitrary fixed element of $\mathfrak{X}$. Let B^0 be the set of those $x \in S$ for which there exists a $\varrho < \omega_{\alpha+n}$ such that $x = x_\varrho^0$. It is obvious that $B^0 \cap F_\varrho \neq 0$ for every $\varrho < \omega_{\alpha+n}$, hence by (4) we have for a $\varrho_0 < \omega_{\alpha+n}$

$$\overline{\overline{B^0 \cap F_{\varrho_0}}} \geqq (r_0 - 1)(n+1) + 1.$$

Put $(r_0-1)(n+1)+1 = k_0$. Then there exists a sequence $\varrho_1^0 < \dots < \varrho_{k_0}^0$ such that $\varrho_0 = \varrho_{i_0}^0$ for an i_0 $(1 \leqq i_0 \leqq k_0)$, $\overline{\overline{\{x_{\varrho_i^0}^0\}}}_{1 \leqq i \leqq k_0} = k_0$ and $\{x_{\varrho_i^0}^0\}_{1 \leqq i \leqq k_0} \subseteq F_{\varrho_0} = F_{\varrho_{i_0}^0}$.

But this means that $B_{\varrho_1^0 \dots \varrho_{k_0}^0}((x_\varrho^0)_{\varrho < \omega_{\alpha+n}}) \cap F_{\varrho_{i_0}^0} = k_0 \not< (r_0-1)n+2$ and thus by (6) $(x_\varrho^0)_{\varrho < \omega_{\alpha+n}} \notin X_{\varrho_1^0 \dots \varrho_{k_0}^0}$ which proves that the product considered in (7) is empty.

Proof of (8). Let $I(\mathfrak{M}')$ denote the set of ordinal numbers ϱ appearing as a subscript ϱ_i $(i = 1, \dots, k)$ of an $X_{\varrho_1 \dots \varrho_k} \in \mathfrak{M}'$. It is obvious that $X_{\varrho_1 \dots \varrho_k} \neq \neq X_{\varrho_1' \dots \varrho_k'}$ if the sequences $\varrho_1, \dots, \varrho_k$ and $\varrho_1', \dots, \varrho_k'$ are different. Hence $\mathfrak{M}' \subseteq \mathfrak{M}$, $\overline{\overline{\mathfrak{M}'}} < \aleph_{\alpha+n}$ implies $\overline{\overline{I(\mathfrak{M}')}} < \aleph_{\alpha+n}$. Thus it is sufficient to see that

$$\bigcap_{\varrho_i (i=1,\dots,k),\, \varrho_i < \varrho_0} X_{\varrho_1 \dots \varrho_k} \neq 0 \quad \text{holds for every} \quad \varrho_0 < \omega_{\alpha+n}.$$

Put $\mathfrak{F}_{\varrho_0} = \{F_{\varrho'}\}_{\varrho' < \varrho_0}$ for every $\varrho_0 < \omega_{\alpha+n}$. Then $p(\mathfrak{F}_{\varrho_0}) = \aleph_\alpha$ by (1). $\mathfrak{F}_{\varrho_0}$ possesses property $\mathbf{C}(2, r_0)$ by (3) and $\overline{\overline{\mathfrak{F}_{\varrho_0}}} \leqq \aleph_{\alpha+n-1}$ $(n-1 \geqq 0)$ for every $\varrho_0 < \omega_{\alpha+n}$. Thus by Theorem 8a) there exists a set B such that

$$1 \leqq \overline{\overline{B \cap F_{\varrho'}}} < (r_0-1)n+2 \quad \text{for every} \quad \varrho' < \varrho_0.$$

It results that we can point out an element $x_{\varrho'}^0$ of $B \cap F_{\varrho'}$ for every $\varrho' < \varrho_0$ and let $x_{\varrho'}^0$ be an arbitrary element of $F_{\varrho'}$ for $\varrho' \geqq \varrho_0$. It is obvious from (6) that the sequence $(x_\varrho^0)_{\varrho < \omega_{\alpha+n}}$ so defined is an element of the product in question.

PROOF OF THEOREM 12. Let $\mathfrak{F}$ be a family, $\overline{\overline{\mathfrak{F}}} = \aleph_\alpha$ such that $\overline{\overline{F}} < \aleph_\alpha$ for every $F \in \mathfrak{F}$. Let $\mathfrak{F} = \{F_\nu\}_{\nu < \omega_\alpha}$ be a well-ordering of type ω_α of $\mathfrak{F}$. Put

$\mathfrak{F}_\nu = \{F_\mu\}_{\mu<\nu}$. Let $\mathfrak{X}$ and $\mathfrak{X}_\nu$ denote the topological product of the elements of $\mathfrak{F}$ and $\mathfrak{F}_\nu$, respectively. If $\Theta = (x_\nu^0)_{\nu<\omega_\alpha}$ is an element of $\mathfrak{X}$, then Θ/ν denotes the element $(x_\mu^0)_{\mu<\nu}$ of $\mathfrak{X}_\nu$.

Let there be given a family $\mathfrak{M}$ of closed subsets of $\mathfrak{X}$. Corresponding to every $X \in \mathfrak{M}$ and $\nu<\omega_\alpha$ we define a subset $Y(X,\nu)$ of $\mathfrak{X}_\nu$ as follows:

$$Y(X,\nu) = \{\Theta/\nu\}_{\Theta\in X}.$$

The set $\{Y(X,\nu)\}_{X\in\mathfrak{M}}$ is of power less than $\aleph_\alpha$ for every $\nu<\omega_\alpha$, since $\aleph_\alpha$ is strongly inaccessible and $\overline{\overline{\mathfrak{X}}}_\nu < \aleph_\alpha$ for every $\nu<\omega_\alpha$. As an easy consequence of this we obtain that $\bigcap_{X\in\mathfrak{M}} Y(X,\nu) \neq 0$ for every $\nu<\omega_\alpha$, provided $\bigcap_{X\in\mathfrak{M}} X \neq 0$ for every $\mathfrak{M}' \subseteq \mathfrak{M}$, $\overline{\overline{\mathfrak{M}'}} < \aleph_\alpha$.

Put $Z_\nu = \bigcap_{X\in\mathfrak{M}} Y(X,\nu)$. The Z_ν's form a ramification system. By a result of P. Erdős and A. Tarski[17] it follows from the hypothesis (**) that there exists a $\Theta \in \mathfrak{X}$ such that $\Theta/\nu \in Z_\nu$ for every $\nu<\omega_\alpha$.

Let X be an arbitrary element of $\mathfrak{M}$. Then for an arbitrary $\nu<\omega_\alpha$ there exists a $\Theta_\nu \in X$ such that $\Theta_\nu/\nu = \Theta/\nu$. Since X is closed, it follows that $\Theta \in X$, and so $\Theta \in \bigcap_{X\in\mathfrak{M}} X$, i. e. $\mathfrak{X}$ is α-compact.

Now we state some unsolved problems which all would have been consequences of $\boldsymbol{T}(\aleph_2, 1) \to 2$. The answer to all these questions is very likely negative, but we can not disprove any of them. In the formulation of all these problems we consider (*) to be assumed.

Problem 6. Let $\mathfrak{F}$ be a family ($\overline{\overline{\mathfrak{F}}} = \aleph_2$, $p(\mathfrak{F}) = \aleph_0$) such that every $\mathfrak{F}' \subseteq \mathfrak{F}$ ($\overline{\overline{\mathfrak{F}'}} \leqq \aleph_1$) possesses property **B**. Does then $\mathfrak{F}$ necessarily possess property **B** too?[18]

The family $\mathfrak{F}$ is said to possess property **G** if there exists a function $f(F)$ defined for every $F\in\mathfrak{F}$ such that $f(F)$ is an element of F and $f(F_1) \neq f(F_2)$ for $F_1 \neq F_2$.

Problem 7. Let $\mathfrak{F}$ be a family ($\overline{\overline{\mathfrak{F}}} = \aleph_2$, $p(\mathfrak{F}) = \aleph_0$) such that every $\mathfrak{F}' \subseteq \mathfrak{F}$ ($\overline{\overline{\mathfrak{F}'}} \leqq \aleph_1$) possesses property **G**. Does then $\mathfrak{F}$ necessarily possess property **G** too?[19]

[17] See the footnote [4] on p. 328 of [8].

[18] The following theorem is an easy consequence of Tychonov's theorem: If $\mathfrak{F}$ is a family of finite sets such that every finite subfamily of $\mathfrak{F}$ possesses property **B**, then $\mathfrak{F}$ possesses property **B**.

[19] This problem is due to W. Gustin (oral communication). It is well known and an easy consequence of Tychonov's theorem that if for a family $\mathfrak{F}$ of finite sets every finite subfamily of it possesses property **G**, then the whole family possesses property **G** too. See e. g. [9].

PROBLEM 8. Let there be given a graph G of power $\aleph_2$. Suppose that every subgraph $\overline{\overline{G}}_1 \leqq \aleph_1$ of G has chromatic number not greater than $\aleph_0$. Is it then true that the chromatic number of G is not greater than $\aleph_0$?[20]

Now we would like to formulate a problem which does not seem to follow directly from $\boldsymbol{T}(\aleph_2, 1) \to 2$, but which belongs to this class of problems too.

PROBLEM 9. Let there be given a graph G of power $\aleph_2$. Suppose that the edges of every subgraph G_1 of G can be directed so that the number of edges emanating from an arbitrary vertex is finite, provided $\overline{\overline{G}}_1 \leqq \aleph_1$.

Is it true that the same holds for the graph G?[21]

A positive solution of Problem 9 would follow from the following generalization of TYCHONOV's theorem. (This generalization is probably false, but as far as we know has not yet been disproved.)

PROBLEM 10. Let $\mathfrak{F}$ be a family of *finite* sets, $\overline{\overline{\mathfrak{F}}} = \aleph_2$, and let $\mathfrak{F} = \{F_\nu\}_{\nu<\omega_2}$ be a well-ordering of type ω_2 of $\mathfrak{F}$. Let $\mathfrak{X}$ denote the Descartes product of the elements of $\mathfrak{F}$, i. e. $\mathfrak{X}$ is the set of all sequences $(x_\nu)_{\nu<\omega_2}$, $x_\nu \in F_\nu$. A subset X of $\mathfrak{X}$ is said to be $\aleph_0$-modified if there exists a set I of ordinal numbers less than ω_2, $\overline{\overline{I}} \leqq \aleph_0$ such that $x_\nu^1 = x_\nu^2$ for every $\nu \in I$ implies that $(x_\nu^1)_{\nu<\omega_2}$ belongs to $\mathfrak{X}$ if and only if $(x_\nu^2)_{\nu<\omega_2}$ belongs to X.

Let $\mathfrak{M}$ be a family of $\aleph_0$-modified subsets of $\mathfrak{X}$ and suppose that the intersection of the elements of every subfamily $\mathfrak{M}'$ of $\mathfrak{M}$ is non-empty, provided $\overline{\overline{\mathfrak{M}'}} \leqq \aleph_1$. Is it true that for an arbitrary family $\mathfrak{M}$ satisfying these conditions $\bigcap_{X\in\mathfrak{M}} X \neq 0$?

9. Further problems. Suppose $p < \aleph_0$.[22] The theorem formulated in the footnote[18] on p. 117 or similar considerations show that to clear up all the problems it would be sufficient to determine the values of the symbols $\mathbf{M}(m, p, q, r) \to \mathbf{B}$, $\mathbf{M}(m, p, r) \to \mathbf{B}(s)$ for finite m's, and so we now suppose that m, p, q, r, s are finite. Obviously, if $r = 1$, then the problems become trivial. So the simplest cases when one can find unsolved problems are $q = 2$, $r = 2$.

[20] It is well known that if every finite subgraph of G has chromatic number not exceeding n, then G has chromatic number not exceeding n. See [10].

[21] As an easy application of TYCHONOV's theorem P. ERDŐS and R. RADO proved the following theorem:

If the edges of every finite subgraph of a given graph G can be directed so that the number of edges emanating from an arbitrary vertex is less than a fixed integer n, then the same is true for the graph G.

[22] T. GALLAI pointed out that interesting and perhaps deep questions can be asked concerning the symbols for p less than $\aleph_0$.

One can ask whether $\mathbf{M}(m, p, 2, 2) \to \mathbf{B}$ is true for a $p > 2$ and for every m. The only non-trivial remark concerning this problem is that

$$\mathbf{M}(7, 3, 2, 2) \not\to \mathbf{B}. \tag{9.1}$$

This is shown by the Steiner triplets for $m = 7$.

The simplest unsolved problem here is

PROBLEM 11. Is it true that

$$\mathbf{M}(m, 4, 2, 2) \to \mathbf{B} \quad \text{holds for every} \quad m?$$

We can not even decide whether there exists an integer p_0 such that

$$\mathbf{M}(m, p_0, 2, 2) \to \mathbf{B} \quad \text{holds for every} \quad m.$$

REMARK. The example (9.1) is best-possible in m, i. e. $\mathbf{M}(6, 3, 2, 2) \to \mathbf{B}$ is true and it is interesting that for $m = 6$, $\mathbf{M}(6, 3, 2, 2) \to \mathbf{B}(2)$ is true too. There remain interesting unsolved problems even if we omit the assumption that $\mathfrak{F}$ possesses property $\mathbf{C}(q, r)$ for some q and r.

It is obvious that if m is sufficiently large, then a family $\mathfrak{F}$ with $p(\mathfrak{F}) = p$, $\overline{\overline{\mathfrak{F}}} = m$ has not to possess property $\mathbf{B}$. Let $m(p)$ denote the least integer m for which such a family exists.

We have

$$m(p) \leqq \binom{2p-1}{p}, \tag{9.2}$$

as it is shown by the subsets taken p at a time of a set having $2p-1$ elements.

More generally, one can ask for the least integers m for which there exists a family $\mathfrak{F}$ $(\overline{\overline{\mathfrak{F}}} = m,\ p(\mathfrak{F}) = p)$ which does not possess property $\mathbf{B}(s)$ where $2 \leqq s \leqq p$. Let $m(p, s)$ denote this integer. (Obviously $m(p, p) = m(p)$.) Similarly as in (9. 2) we have

$$m(p, s) \leqq \binom{p+s-1}{p}. \tag{9.3}$$

(9. 1) shows that the estimations (9. 2) and (9. 3) are far to be best-possible already for $p = 3$. The following problem remains open:

PROBLEM 12. What is the order of magnitude of the functions $m(p)$, $m(p, s)$?

Let us now return to the infinite sets. We would like to raise several new problems, most of which are unsolved, which are all connected to a lesser or greater extent to the ones which we considered so far. To save space we will only outline the partial solutions which we have succeeded in obtaining up to the present.

The first of these problems is the following:

(9.4) Let there be given a family $\mathfrak{F}$ ($\overline{\overline{\mathfrak{F}}}=m, p(\mathfrak{F})=p$) such that every subfamily $\mathfrak{F}'$ of $\mathfrak{F}$ possesses property $\mathbf{B}(r)$, provided that $\overline{\overline{\mathfrak{F}'}}<m$. Under what conditions for the cardinal numbers m, p, r and s does then $\mathfrak{F}$ necessarily possess property $\mathbf{B}(s)$ or property $\mathbf{B}$?

For the sake of brevity we introduce the symbols $\mathbf{S}(m,p,r)\rightarrow\mathbf{B}(s)$, $\mathbf{S}(m,p,r)\rightarrow\mathbf{B}$ ($\mathbf{S}(m,p,r)\not\rightarrow\mathbf{B}(s)$, $\mathbf{S}(m,p,r)\not\rightarrow\mathbf{B}$) to indicate the positive (negative) solutions of the problems, respectively. It is obvious that the problem stated in (9.4) is closely connected with the possible generalizations of TYCHONOV's theorem treated in Section 8. We point out only the simplest and typical problems. A general discussion of this symbol seems to be hopeless at present.

The example given by MILLER cited in Theorem 3 shows, if we assume (*), that

(*)(9.5) $$\mathbf{S}(\aleph_1,\aleph_0,2)\not\rightarrow\mathbf{B}.$$

This follows from the fact that the system of almost disjoint sets of power $\aleph_1$ constructed by MILLER has the following property: if x is an element of the basic set and $S(x)$ is the union of the sets belonging to the system containing x and F is a set of the system not containing x, then $\overline{\overline{S(x)\cap F}}<\aleph_0$.

Comparing Theorems 8 and 10 we obtain as a corrollary that

(*)(9.6) $$\mathbf{S}(\aleph_2,\aleph_0,4)\not\rightarrow\mathbf{B}(4).$$

The following problems remain open:

PROBLEM 13. a) $\mathbf{S}(\aleph_2,\aleph_0,2)\rightarrow\mathbf{B}(2)$ or $\mathbf{S}(\aleph_2,\aleph_0,2)\not\rightarrow\mathbf{B}$?

b) $\mathbf{S}(\aleph_2,\aleph_0,4)\rightarrow\mathbf{B}(5)$ or $\mathbf{S}(\aleph_2,\aleph_0,2)\not\rightarrow\mathbf{B}$?

The following problem concerning the symbol introduced in (9.4) is the simplest one for which our theorems proved so far do not give any information.

PROBLEM 14. Let r be an integer $r\geqq 2$. Is it true that $\mathbf{S}(\aleph_\omega,\aleph_0,r)\rightarrow\mathbf{B}(r)$ holds?

REMARK. It is easy to see that a negative solution of Problem 14 for any r would imply a negative solution of Problem 4.

The second question which arises concerning property $\mathbf{B}$ is the following: Theorem 3 (MILLER's example) assures that there exists a family $\mathfrak{F}$ ($\overline{\overline{\mathfrak{F}}}=2^{\aleph_0}$, $p(\mathfrak{F})=\aleph_0$) such that $\mathfrak{F}$ possesses property $\mathbf{C}(2,\aleph_0)$, but it does not possess property $\mathbf{B}$. However, his example is such that $(\overline{\overline{\mathfrak{F}}})=\aleph_0$ and one can ask whether this is an essential restriction.

Concerning this question, using (*), we can prove the following theorem:

(*) (9.7) *There exists a family* $\mathfrak{F}$ $(\overline{\overline{\mathfrak{F}}}=\aleph_1,\ p(\mathfrak{F})=\aleph_0)$ *which possesses property* $\mathbf{C}(2,\aleph_0)$ *such that it does not possess property* **B** *and satisfies the following condition:*

($\triangle$) $\overline{\overline{(\mathfrak{F}')}}=\aleph_1$ *for every* $\mathfrak{F}'\subseteq\mathfrak{F}$, $\overline{\overline{\mathfrak{F}'}}=\aleph_1$.

We only outline the construction.

Let S be a set, $\overline{\overline{S}}=\aleph_1$. Applying Lemma 3 stated in Section 4 we obtain that there exists a system $\mathfrak{S}$ of subsets of S satisfying the following conditions:

(1) $p(\mathfrak{S})=\aleph_0$, $\overline{\overline{\mathfrak{S}}}=\aleph_1$.

(2) $\mathfrak{S}$ possesses property $C(2,\aleph_0)$.

(3) For an arbitrary $S'\subseteq S$ $(\overline{\overline{S'}}=\aleph_1)$ there exists an $A\in\mathfrak{S}$ such that $A\subseteq S'$.

Let $\mathfrak{S}=\{A_\nu\}_{\nu<\omega_1}$ and $S=\{x_\mu\}_{\mu<\omega_1}$ be well-orderings of type ω_1 of the sets $\mathfrak{S}$ and S, respectively.

Let $\mathfrak{S}_\nu$ be a system of subsets of A_ν for which $p(\mathfrak{S}_\nu)=\aleph_0$, $\overline{\overline{\mathfrak{S}_\nu}}=\aleph_1$, further let $\mathfrak{S}_\nu$ possess property $\mathbf{C}(2,\aleph_0)$. Let $\mathfrak{S}_\nu=\{B^\nu_\mu\}_{\mu<\omega_1}$ be a well-ordering of type ω_1 of the set $\mathfrak{S}_\nu$ for every $\nu<\omega_1$. It is obvious that one can define a monotone increasing sequence $\{\mu_\nu\}_{\nu<\omega_1}$ of type ω_1 of ordinal numbers less than ω_1 such that $\mu_\nu>\mu'$ for every $x_{\mu'}\in A_\nu$ (hence for every $x_{\mu'}\in B^\nu_\mu$ for every $\mu<\omega_1$).

Put $C^\nu_\mu=B^\nu_\mu\cup\{x_{\mu_\nu+\mu}\}$ and $\mathfrak{F}=\{C^\nu_\mu\}_{\nu<\omega_1,\,\mu<\omega_1}$. It is obvious from (1) and (2) that $\overline{\overline{\mathfrak{F}}}=\aleph_1$, $p(\mathfrak{F})=\aleph_0$ and $\mathfrak{F}$ possesses property $\mathbf{C}(2,\aleph_0)$. The fact that $\mathfrak{F}$ does not possess property **B** follows from the property of $\mathfrak{S}$ stated in (3) (taking into consideration that a set which intersects every element of $\mathfrak{F}$ has to be of power $\aleph_1$). Finally, it is easy to verify that if $\overline{\overline{\mathfrak{F}'}}=\aleph_1$, then $\overline{\overline{(\mathfrak{F}')}}=\aleph_1$ for every $\mathfrak{F}'\subseteq\mathfrak{F}$, since if $\overline{\overline{\mathfrak{F}'}}=\aleph_1$, then $\mathfrak{F}'$ either contains $\aleph_1$ C^ν_μ's with the same ν or $\aleph_1$ C^ν_μ's with pairwise different ν's.

The following refinement of the problem solved in (9.7) seems to be interesting. Let us say that the set X is almost contained in Y if $Y-X$ is finite.

Problem 15. Let $\mathfrak{F}$ be a family $(p(\mathfrak{F})=\aleph_0,\ \overline{\overline{\mathfrak{F}}}=\aleph_1)$ such that $\mathfrak{F}$ possesses property $\mathbf{C}(2,\aleph_0)$ and suppose that (instead of ($\triangle$)) it possesses the following property:

At most $\aleph_0$ sets belonging to $\mathfrak{F}$ are almost contained in a denumerable set.

Does such a family $\mathfrak{F}$ necessarily possess property **B**?

The answer is probably negative to this question too, but we can not disprove it even if we omit the assumption that $\mathfrak{F}$ consists of almost disjoint sets.

The following question is connected with Problem 3 (namely a positive solution of it would imply a positive solution of Problem 3b)):

PROBLEM 16. Put $S=\{\nu\}_{\nu<\omega_{\omega+1}}$ $(\overline{\overline{S}}=\aleph_{\omega+1})$.

Let S_ν denote the set $\{\mu\}_{\mu<\nu}$ for every $\nu<\omega_{\omega+1}$. Then $\overline{\overline{S_\nu}}\leqq\aleph_\omega$, and so one can define a splitting of S_ν onto the sum of $\aleph_0$ disjoint sets such that

$$S_\nu=\bigcup_{n<\omega}S_n^\nu \quad \text{and} \quad \overline{\overline{S_n^\nu}}<\aleph_\omega \quad \text{for every} \quad \nu<\omega_{\omega+1}.$$

Is it possible to define the sets S_n^ν in such a way that for every $\nu<\omega_{\omega+1}$ of the second kind which is not cofinal with ω, there exists a monotone increasing sequence $\{\nu_\tau\}_{\tau<\varphi}$ of type φ of ordinal numbers less than ν cofinal with ν and such that $S_n^{\nu_\tau}\subseteq S_n^{\nu_{\tau'}}$ for every n and for every $\tau<\tau'<\varphi$?

A similar but simpler problem is the following one:

PROBLEM 17. Let S be the set of ordinal numbers less than ω_1. Is it possible to define a function $f(\nu)$ on S such that $f(\nu)\in S$, $f(\nu)<\nu$ for every $\nu<\omega_1$ which has the following property: If $\nu<\omega_1$ and ν is of the second kind, then there exists a sequence $\{\nu_n\}_{n<\omega}$ of type ω of ordinal numbers less than ν such that $\nu_n\to\nu$ and $f(\nu_{n+1})=\nu_n$ for $n=0,1,2,\ldots$. This problem is interesting in itself and seems to be very difficult.

The positive solution of the following problem would imply a negative solution of an immediate generalization of Problem 9, namely it would assure the existence of a graph G of power $\aleph_{\omega+1}$ the edges of every subgraph of power $\aleph_1$ of which can be directed so that the number of edges emanating from a vertex should be finite, but the whole graph can not be directed in such a way.

PROBLEM 18. Let S be a set of power $\aleph_\omega$. Does there exist a family $\mathfrak{F}$ such that $(\mathfrak{F})\subseteq S$, $\overline{\overline{\mathfrak{F}}}=\aleph_{\omega+1}, p(\mathfrak{F})=\aleph_0$, and which has the following property:

(1) If $S'\subseteq S$, $\overline{\overline{S'}}\leqq\aleph_0$, then there exist at most $\aleph_0$ sets F belonging to the family such that $\overline{\overline{F\cap S'}}=\aleph_0$.

REMARK. On the one hand, we can not disprove Problem 18 even if we require that $\mathfrak{F}$ should possess property $\mathbf{C}(2,\aleph_0)$, on the other hand we can not prove it if we require only that $\mathfrak{F}$ should possess the following weaker property instead of (1):

Every $S'\subseteq S$ $(\overline{\overline{S'}}=\aleph_0)$ contains at most $\aleph_0$ elements of the family.

We construct the graph G mentioned above as follows: Suppose that the family $\mathfrak{F}$ and the set S satisfy the requirement of Problem 18. Let the set of vertices of G be $\mathfrak{F} \cup S$. The edges are the pairs (F, x) where $F \in \mathfrak{F}$ and $x \in F$. It is easy to see that G has the property required.

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Added in proof (MARCH 3, 1961). The manuscript of this paper had been written before the authors knew that A. TARSKI has disproved the hypothesis (**). (See A. TARSKI, Some problems and results relevant to the foundations of set theory, *Proceedings of the International Congress for Logic, Methodology and Philosophy of Science* (Stanford, 1960).)

Thus we have no arguments to prove our Theorem 12 proved with the help of this hypothesis. It seems that the theorem is false at least for the inaccessible cardinals m which are strongly incompact.

It is obvious that the discussion of the unsolved problems concerning the symbol $\boldsymbol{T}(m, \lambda) \rightarrow \varkappa$ has to be changed in some places knowing the new result of A. TARSKI.

References

[1] E. W. MILLER, On a property of families of sets, *Comptes Rendus Varsovie*, **30** (1937), pp. 31—38.

[2] A. TARSKI, Sur la décomposition des ensembles en sous ensembles presque disjoint, *Fundamenta Math.*, **14** (1929), pp. 205—215.

[3] J. ŁOS, Linear equations and pure subgroups, *Bull. Acad. Polon. Sci. Math.*, **7** (1959), pp. 13—18.

[4] F. BERNSTEIN, Zur Theorie der trigonometrischen Reihen, *Leipz. Ber.*, **60** (1908), pp. 325—338.

[5] P. ERDŐS and A. HAJNAL, On the structure of set mappings, *Acta Math. Acad. Sci. Hung.*, **9** (1958), pp. 111—131.

[6] A. HAJNAL, Some problems and results on set theory, *Acta Math. Acad. Sci. Hung.*, **11** (1960), pp. 277—298.

[7] S. ULAM, Zur Maßtheorie in der allgemeinen Mengenlehre, *Fundamenta Math.*, **16** (1930), pp. 140—150.

[8] P. ERDŐS and A. TARSKI, On families of mutually exclusive sets, *Annals of Math.*, **44** (1943), pp. 315—329.

[9] P. R. HALMOS and H. E. VAUGHAN, The marriage problem, *Amer. Journ. Math.*, **72** (1950), pp. 214—215.

[10] N. G. DE BRUIJN and P. ERDŐS, A colour problem for infinite graphs and a problem in the theory of relations, *Indag. Math.*, **13** (1951), pp. 371—373.