A graph is said to be $k$-chromatic if its vertices can be split into $k$ classes so that two vertices of the same class are not connected (by an edge) and such a splitting is not possible for $k-1$ classes. Tutte was the first to show that for every $k$ there is a $k$-chromatic graph which contains no triangle [1].

The lower girth of a graph is defined as the smallest integer $t$ so that our graph has a circuit of $t$ edges. J. B. Kelly and L. M. Kelly [2] showed that there exist graphs of arbitrarily high chromatic number and lower girth 6. I proved [3] that for every $t$ and $k$ there is a graph of chromatic number $k$ and lower girth $t$. In fact I showed the following sharper result: To every $k$ there is an $\epsilon$ so that for $n > n_0(\epsilon, k)$ there is a $G^{(n)}$ ($G^{(n)}$ will denote a graph of $n$ vertices, $G_l^{(n)}$ will denote a graph with $n$ vertices and $l$ edges) of chromatic number $k$ and lower girth $\geq \epsilon \log n$.

We shall show that in some sense this result is best possible. First we introduce some notations. $f(m, k; n)$ denotes the maximum of the chromatic number of all graphs $G^{(n)}$, every subgraph of $m$ vertices of which has chromatic number not exceeding $k$; $g_k(n)$ is the largest integer for which there is a $G^{(n)}$ of chromatic number $k$ and lower girth $g_k(n)$. Clearly $g_3(n)$ is the largest odd integer not exceeding $n$ (since every odd circuit has chromatic number 3). For $k > 3$ the determination of $g_k(n)$ seems very difficult. In [3] I proved† ($c_1, c_2, \ldots$ will denote suitable positive constants)

$$g_k(n) > c_1 \frac{\log n}{\log k}.$$  \hfill (1)

Now I shall prove

**Theorem 1.** For $k \geq 4$ we have

$$g_k(n) \leq \frac{2 \log n}{\log (k-2)} + 1.$$  \hfill (2)

Theorem 1 and (1) shows that for $k \geq 4$ the order of magnitude of $g_k(n)$ is $\log n$ (it would be easy to replace (1) by an explicit inequality). It seems likely that for $k > 3$

$$\lim_{n \to \infty} \frac{g_k(n)}{\log n}$$

exists, but I have not been able to prove this.

Theorem 1 shows that the chromatic number can be "large" only if the lower girth is $\leq \epsilon \log n$. Theorem 1 further implies that every $G^{(n)}$

† In [3], (1) is proved in a slightly different form.

(MATHEMATICA 9 (1962), 170-175)
which is 4-chromatic must contain a circuit of length $\leq 1 + 2 \log_2 n$. I thought that every 4-chromatic $G^{(n)}$ must also contain an odd circuit of length $< c_2 \log n$. In other words, I conjectured that for a sufficiently large constant $c_2$ we have $f([c_2 \log n], 2; n) = 3$ (a graph all of whose circuits are even is 2-chromatic). T. Gallai (not knowing of my conjecture) constructed a 4-chromatic $G^{(n)}$ the smallest odd circuit of which has length $[n^2]$. Gallai's example is not yet published. Gallai and I then conjectured that the largest value of $m$ for which $f(m, 2; n) = k$ is of the order of magnitude $n^{1/(k+2)}$, but we have not even been able to prove that for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$, $f([c_2 \log n], 2; n) = 3$.

The situation seems to change quite radically if we consider $f(m, 3; n)$ instead of $f(m, 2; n)$. In fact I shall prove

**Theorem 2.** To every $k$ there is an $\varepsilon > 0$ so that if $n > n_0(\varepsilon, k)$ there exists a $k$-chromatic $G^{(n)}$ every subgraph of which having $[en]$ vertices is at most 3 chromatic.

Instead of Theorem 2 we shall prove the following stronger

**Theorem 3.** For $m > 3$ we have

$$f(m, 3; n) > c_3 \left( \frac{n}{m} \right)^{1/3} \left( \log \frac{n}{2m} \right)^{-1}.$$  

(1')

For $f(m, k; n)$ at present we only can show a trivial upper bound:

$$f(m, k; n) \leq \left[ \frac{n}{m} + 1 \right] k.$$  

(2)

(2) is indeed trivial since we can split the vertices of $G^{(n)}$ into at most $[n/m] + 1$ sets each having $\leq m$ elements, and by assumption the graphs spanned by these vertices are at most $k$-chromatic.

(2) is certainly very far from being best possible. It is easy to deduce from a result of Szekeres and myself [4] that for $n > k$ $f(m, k, n)$ in fact is meaningful only for $m > k$

$$f(m, k; n) \leq f(k+1, k; n) < c_4 n^{1-\left(\frac{1}{k+1}\right)}.$$  

(3)

The deduction of (3) from [4] is easy and can be left to the reader (to simplify his task we only remark that if every subgraph of $k+1$ vertices of $G^{(n)}$ is at most $k$-chromatic then $G^{(n)}$ cannot contain a complete $(k+1)$-gon $G^{(k+1)}$).

I further proved that [5]

$$f(3, 2; n) > c_5 n^{1/\log n}.$$  

(4)

It seems probable that

$$f(k+1, k; n) > n^{1-\left(\frac{1}{k+1}\right)^e},$$  

(5)
for every \( \epsilon > 0 \) if \( n > n_0(\epsilon, k) \). I do not know to what extent the exponent \( \frac{1}{2} \) in Theorem 3 can be improved for all values of \( m \).

**Proof of Theorem 1.** A simple induction argument shows that every \( k \)-chromatic \( G^{(m)} \) contains a subgraph \( G^{(m)} \) every vertex of which has valency \( \geq k - 1 \) (the valency, or order, of a vertex is the number of edges incident to it). Assume now that \( G^{(m)} \) is \( k \)-chromatic and is of lower girth \( t \). Let \( G^{(m)} \) be a subgraph of \( G^{(m)} \) every vertex of which has valency \( \geq k - 1 \) and let \( X_0 \) be any vertex of \( G^{(m)} \). Consider the set of vertices of \( G^{(m)} \) which can be reached from \( X_0 \) by a path of \([t - 1]/2 \) or fewer edges. Clearly every such vertex can be reached by only one such path (for otherwise \( G^{(m)} \), and therefore \( G^{(m)} \), would contain a circuit of fewer than \( t \) edges). Since, further, every vertex of \( G^{(m)} \) has valency \( \geq k - 1 \), we obtain by a simple argument that there are more than \((k - 2)^{[t - 1]/2} \) vertices which can be reached from \( X_0 \) by a path of \([t - 1]/2 \) or fewer edges. Hence

\[
(k - 2)^{[t - 1]/2} \leq (k - 2)^{[t - 1]/2} \leq n,
\]

which proves Theorem 1.

The proof of Theorem 3 will use simple probabilistic arguments and will be similar to previous proofs used by Renyi and the author [5]. First we need two Lemmas which are of independent interest. Denote by \( G^{(m)} \) a graph having \( n \) vertices and \( l \) edges. If the vertices are labelled then the number of different graphs \( G^{(m)} \) clearly equals \( (\begin{pmatrix} n \\ l \end{pmatrix}) \). A set of vertices of \( G^{(m)} \) is said to be independent if no two of them are connected by an edge.

**Lemma 1.** Let \( l = [rn] \), \( r > \epsilon_0 \); then for all except possibly \( o(r) \) graphs \( G^{(m)} \) the maximum number of independent vertices is less than \((n/r) \log r \).

Let \( x_1, \ldots, x_n \) be the vertices of \( G^{(m)} \). The number of graphs \( G^{(m)} \) for which \( x_1, \ldots, x_n \) is an independent set is clearly

\[
\left( \begin{pmatrix} n \\ l \end{pmatrix} - \begin{pmatrix} l \end{pmatrix} \right).
\]

Since the vertices can be chosen in \( \begin{pmatrix} n \\ l \end{pmatrix} \) ways, the number of graphs \( G^{(m)} \) for which the maximum number of independent points is \( \geq u \) is not greater than

\[
\begin{align*}
\binom{n}{u} \left( \frac{\begin{pmatrix} n \\ l \end{pmatrix} - \begin{pmatrix} l \end{pmatrix}}{l} \right) &< \frac{n^u u^u}{u^u} \left( \frac{\begin{pmatrix} n \\ l \end{pmatrix} - \begin{pmatrix} l \end{pmatrix}}{l} \right) < \frac{e^u}{u^u} \left( 1 - \frac{\begin{pmatrix} n \\ l \end{pmatrix}}{l} \right)^l \left( \frac{n}{l} \right) \\
&< \left( \frac{n}{l} \right) \frac{e^u}{u^u} e^{-iu^u}. \tag{5}
\end{align*}
\]

† This idea is used in [3] and also in Lemma 3 of P. Erdős and L. Pósa. "On the maximal number of disjoint circuits of a graph", *Publ. Math. Debrecen*, 9 (1962), 3–12.
By (5) the proof of our lemma will be complete, if we show that, for $u \geq (n/r) \log r$, $r > c_6$, we have

$$\left(\frac{en}{u}\right)^u e^{-u^2/n^2} < \frac{1}{10}. \quad (6)$$

(6) can be shown by a simple computation and is left to the reader.

It would be easy to drop the condition $r > c_6$, but then $(n/r) \log r$ would have to be replaced by, say,

$$\frac{n \log (r+2)}{r+c_7}. \quad (7)$$

It seems that the order of magnitude $(n/r) \log r$ is not far from being best possible at least for certain ranges of $r$.

**Corollary.** Let $l = \lceil rn \rceil$, $r > c_6$. Then for all except $10^{-15}$ graphs $G_i^{(n)}$ the chromatic number of $G_i^{(n)}$ is greater than $r \log r$.

The corollary immediately follows from Lemma 1, since if $G_i^{(n)}$ is $k$-chromatic the maximum number of independent vertices must be $\geq n/k$ (since the $n$ vertices can be split into $k$ independent sets).

**Lemma 2.** Let $l = \lceil rn \rceil$, $r > c_6$. Then for all but $10^{-15}$ graphs $G_i^{(n)}$ every subgraph spanned by $u$ of its vertices, $4 \leq u \leq 10^{-6}nr^{-3}$, contains fewer than $\frac{u}{2}$ edges.

In particular the lemma implies that these $G_i^{(n)}$ contain no complete quadrilateral. This result is contained in my paper with Rényi quoted in [6].

Denote by $N(u, t)$, $4 \leq u \leq 10^{-6}nr^{-3}$, $\frac{u}{2} \leq t \leq \min\left(\binom{u}{2}, l\right)$ the number of graphs $G_i^{(n)}$ which contain a subgraph $G_i^{(u)}$. To prove our lemma we have to show that

$$\sum_u \sum_t N(u, t) < \frac{1}{10} \binom{\binom{u}{2}}{l}, \quad (7)$$

where the summation is extended over $4 \leq u \leq 10^{-6}nr^{-3}$,

$$\frac{u}{2} \leq t \leq \min\left(\binom{u}{2}, l\right).$$

First we estimate $N(u, t)$. Let $x_{i_1}, \ldots, x_{i_u}$ be any $u$ vertices of $G_i^{(n)}$. The number of graphs $G_i^{(n)}$ for which the subgraph spanned by $x_{i_1}, \ldots, x_{i_u}$ contains $t$ edges clearly equals

$$\binom{\binom{u}{2}}{t} \binom{\binom{u}{2} - \binom{u}{2}}{l-t}. \quad (8)$$

Since the vertices $x_{i_1}, \ldots, x_{i_u}$ can be chosen in $\binom{n}{u}$ ways, we evidently have

$$N(u, t) = \binom{n}{u} \binom{\binom{u}{2} - \binom{u}{2}}{l-t}. \quad (9)$$
From (8) we obtain by a simple computation

\[
(t > \frac{2}{\log n}, l = [rn], u \leq 10^{-6}nr^{-3}).
\]

\[
N(u, t)\left(\left(\begin{array}{c} n \\ l \end{array}\right)\right) - \frac{u^2}{n^2} < \left(\frac{e^{-2}}{u} \right) t < \left(\frac{e^{23}e^{3l}}{u} \right)
\]

\[
< \left(\frac{10u^{1/3}l^3}{n^{4/3}}\right)^{t} \leq \left(\frac{10rn(10^{-6}nr^{-3})^{1/3}}{n^{4/3}}\right)^{t} = 10^{-t}.
\]

From (9) we easily obtain by \(u \geq 4, t \geq \frac{2}{\log n}\), that (7) holds and hence our lemma is proved. \(r \leq \frac{n^{4/3}}{4^{1/2}.100}\) was needed to make sure that \(10^{-6}nr^{-3} \geq 4\) should be true, in other words, that the range for \(u\) should not be empty.

**Corollary.** Let \(l = [rn] \leq \frac{n^{4/3}}{4^{1/2}.100}\), then for all but \(1 - \frac{1}{10}\left(\begin{array}{c} 2 \\ l \end{array}\right)\) graphs \(G^{(n)}\) every subgraph spanned by \(u\) of its vertices \(u \leq 10^{-6}nr^{-3}\) is at most 3-chromatic.

As stated previously a simple induction argument shows that every \(G^{(n)}\) of chromatic number \(\geq 4\) contains a subgraph \(G^{(o)}\) every vertex of which has valency \(\geq 3\). Thus \(G^{(o)}\) has at least \(\frac{2}{3}u\) edges and the corollary follows from Lemma 2.

The constant \(10^{-6}\) could easily be replaced by a larger one and the exponent \(-3\) in \(10^{-6}nr^{-3}\) could also be slightly increased, but I do not pursue these investigations since the corollary is sharp enough to deduce Theorems 2 and 3 and at present I cannot obtain best possible estimations, or even estimations which are likely to be anywhere near being best possible.

Now we can prove Theorem 3. Put \(r = \frac{1}{100}(n/m)^{1/3}, l = [rn]\). By the corollary to Lemma 1 we first of all obtain that for all but \(\frac{1}{10}\left(\begin{array}{c} 2 \\ l \end{array}\right)\) graphs \(G^{(m)}\) their chromatic number is greater than

\[
\frac{r}{\log r} > c_3\left(\frac{n}{m}\right)^{1/3}\left(\log \frac{2n}{m}\right)^{-1},
\]

if \(c_3\) is sufficiently small. (Lemma 1 applies since we can assume that \(r > c_6\), for if not then \(m \geq 10^{-4}nc_6^{-3}\) and for sufficiently small \(c_3\) (1') becomes trivial.)

Secondly, by the corollary to Lemma 2 (since \(m \geq 4, r \leq \frac{n^{1/3}}{4^{1/2}.100}\) and Lemma 2 applies) for all but \(\frac{1}{10}\left(\begin{array}{c} 2 \\ l \end{array}\right)\) graphs \(G^{(m)}\) the chromatic number of all their subgraphs having at most \(u\) vertices is \(\leq 3\) for

\[
u \leq 10^{-6}nr^{-3} = m,
\]

(11)
(10) and (11) implies that for $m \geq m_0$ at least $\frac{1}{2} \binom{m}{2}$ of the graphs satisfies (1'), which completes the proof of Theorems 3 and 2.

To conclude I just wish to remark that from (4) one can deduce a much stronger result than is obtained by putting $m = 4$ in Theorem 3.

References.


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