REMARKS ON A PAPER OF PÓSA

by

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This note will use the terminology of Pósa's paper. \( G_l(n) \) will denote a graph of \( n \) vertices and \( l \) edges and \( G_l(n,k) \) denotes a graph of having \( n \) vertices \( l \) edges and every vertex of which has valency \( \geq k \). Ore [2] proved that if \( l \geq \binom{n-1}{2} + 2 \) then every \( G_l(n) \) is Hamiltonian, and he showed that the result is false for \( l = \binom{n-1}{2} + 1 \). Now I prove the following more general

Theorem. Let \( 1 \leq k < n/2 \). Put

\[
l_k = 1 + \max_{1 \leq t < \frac{n}{2}} \left( \binom{n-t}{2} + t^2 \right) =
\]

(1)

\[
1 + \max \left( \binom{n-k}{2} + k^2, \left( n - \left\lfloor \frac{n-1}{2} \right\rfloor \right) + \left( \frac{n-1}{2} \right)^2 \right).
\]

Then every \( G_l(n,k) \) is Hamiltonian. There further exists a \( G_{l-1}(n,k) \) which is not Hamiltonian.

First of all observe that by the theorem of Dirac (see the preceding paper of Pósa) if every vertex of \( G \) has valency \( \geq n/2 \) then \( G \) is Hamiltonian, thus the condition \( 1 \leq k < n/2 \) can be assumed without loss of generality.

Next a simple computation shows that \( \binom{n-t}{2} + t^2 \) decreases for \( 1 \leq t \leq (n - 2)/3 \) and increases for \( (n - 2)/3 < t < n/2 \), which proves the second equality of (1).

Now we are ready to prove our Theorem. If our \( G_l(n,k) \) is not Hamiltonian then by the Theorem of Pósa there exists a \( t, k \leq t < n/2 \) so that \( G_l(n,k) \) has at least \( t \) vertices \( x_1, \ldots, x_t \) of valency not exceeding \( t \). The number of edges of \( G_l(n,k) \) which are not incident to any of the vertices \( x_1, \ldots, x_t \) is clearly at most \( \binom{n-t}{2} \) (i.e. if the vertices of \( G_l(n,k) \) are \( x_1, \ldots, x_n \) we obtain \( \binom{n-t}{2} \) edges if every two of the vertices \( x_{j_1} \) and \( x_{j_2}, t < j_1 < j_2 \leq n \) are connected.
by an edge). The number of edges incident to one of the vertices \(x_1, \ldots, x_i\) is at most \(t^2\) (since each of them has valency \(\leq t\)). Thus our \(G_{(n)}(k)\) has at most \(\binom{n-t}{2} + t^2\) edges for some \(k \leq t < n/2\) i.e. it can have at most \(l_k - 1\) edges which proves (1).

To complete our proof we show that (1) is best possible. Let the vertices of \(G_{(n)}(1)\) be \(x_1, \ldots, x_n\). Its edges are:

\[(x_i, x_j), \ t < i < j \leq n \text{ and } (x_i, x_j), \ 1 \leq i \leq t < j \leq 2t < n.\]

A simple argument shows that our \(G_{(n)}(1)\) is not Hamiltonian (it clearly has \(l_1 - 1\) edges). It is easy to see that every \(G_{(n)}(t)\) which is not Hamiltonian has this structure. (If \(t = (n-1)/2\) (n odd) by Pósa’s theorem we can assume that there are \(t + 1 = (n + 1)/2\) vertices of valency \(\leq t\) but by \(\frac{(t+1)^2}{2} + t^2\) we do not obtain a better result by utilising this \((t+1)\)-st vertex).

It is easy to see that the argument of Pósa’s paper gives the following.

**Theorem.** Let \(G_{(n)}\) be a graph and assume that for every \(1 \leq k < (n - 1)/2\) \(G_{(n)}(k)\) has at most \(k\) vertices of valency \(\leq k\). Then \(G_{(n)}\) has an open Hamilton line.

The theorem is best possible.

The proof can be left to the reader of Pósa’s paper. Using this result we obtain by the same argument as used in this paper that every \(G_{(n)}(k)\) with

\[\mu_k = 1 + \max_{k \leq t < \frac{n-1}{2}} \left[\binom{n-t-1}{2} + t(t+1)\right] \]

has an open Hamilton line. The theorem is best possible. Finally we mention that by the method of this paper we can prove the following sharpening of Lemma (3.2) of [1]. Let \(G_{(n)}\) be a graph with the vertices \(x_1, \ldots, x_n\) and \(2 \leq k < n/2\). Assume that \(v(x_i) \geq k\) and that there is a circuit containing the vertices \(x_2, x_3, \ldots, x_n\). Then if \(G_{(n)}\) has \(\geq l_k\) edges it is Hamiltonian. The result is best possible. We leave the simple proof to the reader.

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**REFERENCES**


**Теорема.** Пусть $1 \leq k < \frac{n}{2}$.

$$l_k = 1 + \max_{k \leq t < \frac{n}{2}} \left[ \left( \frac{n - t}{2} \right) + t^2 \right] =$$

$$= 1 + \max \left[ \left( \frac{n - k}{2} \right) + k^2, \left( \frac{n - \left[ \frac{n - 1}{2} \right]}{2} \right) + \left[ \frac{n - 1}{2} \right]^2 \right].$$

Тогда в каждом графе $G$, который имеет $n$ точек и в котором степень каждой точки $\geq k$ и число ребер равно $l_k$, существует Гамильтонова линия, то есть окружность, содержащая все точки $G$. Теорема точна.