1. Introduction. A graph $G$ is called complete if any two of its vertices are connected by an edge; a set of vertices of $G$ are said to be independent if no two of them are connected by an edge. It follows from a well-known theorem of Ramsay (1) that for each pair of positive integers $k, l$ there is an integer $f(k, l)$, which we take to be minimal, such that every graph with $f(k, l)$ vertices either contains a complete graph of $k$ vertices or a set of $l$ independent points. Szekeres (2) proved that

$$f(k, l) \leq \binom{k + l - 2}{k - 1},$$

and Erdős (3; 4) that

$$f(k, k) \geq 2^{k/2},$$

$$f(3, l) > l^{1+c_3},$$

for a positive constant $c_3$.

Clearly

$$f(k, l) \geq f(3, l) > l^{1+c_3},$$

for $k \geq 4$. Our object is to prove a stronger result. We say that a set $S$ of points of a graph $G$ is $m$-independent, if there is no complete subgraph of $G$ having $m$ vertices in $S$. Let $h(k, l)$ be the minimal integer such that every graph of $h(k, l)$ vertices contains either a complete graph of $k$ vertices or a set of $l$ points which are $(k - 1)$-independent. Then clearly

$$h(k, l) \leq f(k, l)$$

for all $k, l$. However we can still prove that

$$h(k, l) > l^{1+c_3},$$

for $k \geq 3$. This problem is due to A. Hajnal (oral communication).

Our construction is geometric, and is based on a lemma (§2) of some geometric interest.

2. Regular simplices on the surface of a sphere. We define the relative surface area of a set $S$ on the surface of a sphere in $n$-dimensional euclidean space to be the surface area of $S$ divided by the surface area of the sphere. We prove

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LEMMA. Suppose $n$ and $k$ are positive integers ($k \leq n$) and that $\zeta$ satisfies

$$0 < \zeta < \sqrt{2},$$
$$k\left[1 - \left(\frac{\zeta}{k}\right)^2\right]^{n/2} < 1.$$ 

Then, if $S$ is a set on the surface of the unit sphere $\Sigma$ in $n$-dimensional space of relative surface area

$$V > \left[1 - \left(\frac{\zeta}{k}\right)^2\right]^{n/2},$$

there is a regular $k$-simplex, with its vertices each on $\Sigma$ within a distance* $\zeta$ of $S$, and with its centre at the centre of $\Sigma$.

Remark. This lemma shows that in a space of many dimensions even a set of rather small relative surface area on the unit sphere will always contain a $k$-simplex, which is very nearly a regular $k$-simplex of unit circum-radius.

Proof. Let $C$ be the minor spherical cap cut from $\Sigma$ by a plane passing at a distance $\frac{\zeta}{k}$ from its centre. Since $\zeta < \sqrt{2}$, it is clear that the union of the segments joining the centre $O$ to the points of $C$ is contained in the sphere with radius

$$\left[1 - \left(\frac{\zeta}{k}\right)^2\right]^{1/2}$$

with its centre at the centre of the base of the cap $C$. Consequently the relative surface area of $C$ is at most

$$\left[1 - \left(\frac{\zeta}{k}\right)^2\right]^{n/2} < V,$$

and so is less than the relative surface area of $S$. Let $C_T$ and $S_T$ be the sets of points on $\Sigma$ within the distance $\zeta$ of the points of $C$ and $S$ respectively. Then by a well-known result of Schmidt (5) the relative surface area of $S_T$ will be at least that of $C_T$. But $C_T$ is the major cap cut from $\Sigma$ by a plane passing at the distance $\frac{\zeta}{k}$ from 0, and so has relative surface area at least

$$1 - \left[1 - \left(\frac{\zeta}{k}\right)^2\right]^{n/2} > 1 - (1/k).$$

So the relative surface area of the set $T$ of points of $\Sigma$ not in $S_T$ is less than $1/k$.

Consider the space $\mathcal{S}_k$ of all ordered sets $X = \{x_1, x_2, \ldots, x_k\}$ of $k$ points of $\Sigma$ forming a regular $k$-simplex of circum-radius 1 with the metric

$$d(X, Y) = \sqrt{\sum_{i=1}^{k} |x_i - y_i|^2}.$$ 

It is possible to introduce a measure on the Borel sets of $\mathcal{S}_k$ giving the whole space unit measure and such that, for $i = 1, 2, \ldots, k$, the measure of the set $\mathcal{X}_i$ of points $X = \{x_1, x_2, \ldots, x_k\}$ with $x_i \in T$ is equal to the relative surface area of $T$ and so is less than $1/k$. Hence we can choose a point $X$ of $\mathcal{S}_k$ not in $\bigcup_{i=1}^{k} \mathcal{X}_i$.

*All our distances are measured in the $n$-dimensional space, not on the surface of the sphere.
The points $x_1, x_2, \ldots, x_k$ form a regular $k$-simplex of circum-radius 1 in $S_t$ and so within distance $\xi$ of $S$. This proves the lemma.

3. **Theorem.** Let $k > 3$ be an integer. If $c_k$ is a positive constant less than

$$\frac{\log 1/\left[1 - \left(\frac{1}{8}\right)^2\right]}{2 \log 4/\eta_k},$$

where

$$1/\eta = 1/\eta_k = \frac{1}{2}(k - 1)^{1/2}(k - 2)^{1/2}[\{2(k - 1)^2\}^{1/2} + \{2k(k - 2)^2\}^{1/2}],$$

and $l$ is a sufficiently large integer, there is a graph $G$, with less than

$l^{1+c_k}$

vertices, which contains no complete $k$-gon, but such that each subgraph with $l$ vertices contains a complete $(k - 1)$-gon.

**Remark.** We can take $c_k \sim 1/(512k^4 \log k)$ as $k \to \infty$.

**Proof.** Let $H$ be the greatest integer less than $l^{1+c_k}$. Let $\epsilon$ be a small positive constant and let $n$ be the nearest integer to

$$(1 + \epsilon)\log H / \log \left[\frac{4}{\eta \sqrt{1 - \left(\frac{1}{8}\right)^2}}\right].$$

We take the vertices of our graph to be a set $N$ of $H$ points on the surface of the sphere $\Sigma$ in euclidean $n$-dimensional space with centre at the origin $O$ and with unit radius, and we join each pair whose distance apart exceeds

$$\sqrt{2k/(k - 1)}.$$

Since the unit sphere contains no simplex with $k$ vertices with all its edges exceeding this length our graph contains no complete $k$-gon. But if $(k - 1)$ points of $N$ have mutual distances apart exceeding

$$\sqrt{2(k - 1)/(k - 2)} - \eta_k = \sqrt{2k/(k - 1)},$$

they will form a complete $(k - 1)$-gon in the graph. Thus to prove the theorem it will suffice to prove that the points of $N$ can be chosen, so that from any set of $l$ points of $N$ a subset of $(k - 1)$ points may be chosen with their mutual distances apart exceeding

$$\sqrt{2(k - 1)/(k - 2)} - \eta.$$

With each point $x$ of $\Sigma$ and each $\xi$ with $0 < \xi < 1$ we associate the spherical cap $C(x, \xi)$ of all points of $\Sigma$ within a distance $\xi$ of $x$. Now the union of the segments joining $O$ to the points of $C(x, \xi)$ contains a cone with $O$ as vertex of height

$$1 - \frac{1}{2} \xi^2.$$
with a \((n - 1)\)-dimensional sphere of radius 
\[ \xi(1 - \frac{1}{4} \xi^2)^{1/2} \]
as its base. But the unit sphere is itself contained in a cylinder of height 2 with a \((n - 1)\)-dimensional unit sphere as its base. Hence the relative surface area of \(C(x, \xi)\) is at least 
\[ \frac{1}{2n} (1 - \frac{1}{4} \xi^2) [\xi(1 - \frac{1}{4} \xi^2)^{1/2}]^{n-1} > \frac{1}{4n} [\xi(1 - \frac{1}{4} \xi^2)^{1/2}]^n. \]
Since \(0 < \eta < 1\) we can choose \(\xi\) with \(0 < \xi < \frac{1}{4}\eta\) so that the relative surface area \(V\) of \(C(x, \xi)\) is exactly 
\[ V = \frac{1}{4n} [\frac{1}{4}\eta(1 - \frac{1}{4}\eta^2)^{1/2}]^n. \]

Let \(S\) be the union of all the caps \(C(x, \xi)\) with \(x\) in \(N\). Let \(h\) be the integer nearest to \(H^*\). Since 
\[
\log (h + 1) - \log \{ (H + 1) V \} = \epsilon \log H - \log H + n \log \left( \frac{1}{\eta} \right) \left( 1 - \frac{1}{4} \eta^2 \right)^{-1} + O(\log n)
\]
we have 
\[ h + 1 > (H + 1) V, \]
provided \(l\) is sufficiently large. A simple probability argument, which we have recently used elsewhere \((6)\), shows that, if the \(H\) points of the set \(N\) are distributed independently uniformly over \(\Sigma\), then the expectation of the relative surface area of the set \(F_h\) of points of \(\Sigma\) which lie in \(h\) or more of the caps 
\(C(x, \xi)\) with \(x\) in \(N\) is at most 
\[ \frac{H!}{h!(H - h)!} V^h (1 - V)^{H-h} \left( \frac{h + 1}{h} - (H + 1)V \right). \]
So we may suppose that the points of \(N\) are chosen so that the relative surface area \(V_h\) of the set \(F_h\) satisfies 
\[ V_h < \frac{H!}{h!(H - h)!} V^h (1 - V)^{H-h} \left( \frac{h + 1}{h} - (H + 1)V \right). \]
Now 
\[ h = H^* + O(1), \]
and 
\[ V = \frac{1}{4n} [\frac{1}{4}\eta(1 - \frac{1}{4}\eta^2)^{1/2}]^n \]

\[ = \exp \left[ - n \log \frac{4}{\eta \sqrt{(1 - \frac{1}{4}\eta^2)}} + O(\log n) \right] \]
\[ = \exp [- (1 + \epsilon) \log H + O(\log \log H)] \]
\[ = \frac{1}{(\log H)^\alpha(1)} H^{-1}. \]
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So, using Stirling's formula and making some elementary reductions, we have

\[ \log V_h - \log \frac{1}{2} V \]
\[ < \log \left[ 2 \frac{H!}{h!(H - h)!} V^{n-1}(1 - V)^{n-h} \frac{(h + 1)(1 - V)}{(h + 1) - (H + 1) V} \right] \]
\[ = - 2eH^2 \log H + O(H^3 \log \log H). \]

Thus \( V_h < \frac{1}{2} V \), when \( l \) is sufficiently large.

Let \( L \) be a subset of \( N \) with \( l \) elements. Let \( C'(x, \xi) \) be the part of \( C(x, \xi) \) not lying in \( F_h \). The relative surface area of \( C'(x, \xi) \) is at least

\[ V - V_h > \frac{1}{2} V. \]

The points of the union \( S_L \) of the sets \( C'(x, \xi) \) with \( x \) in \( L \) belong to at most \( h - 1 \) of the sets \( C'(x, \xi) \). So the relative surface area \( V_L \) of \( S_L \) is at least

\[ \frac{1}{2} Vl/(h - 1). \]

Hence

\[ \log V_L - \log[1 - (\frac{1}{2} \cdot \eta)^2]^{n/2} \]
\[ \geq \log \left[ \frac{1}{2} Vl/(h - 1) \right] - \log[1 - (\frac{1}{2} \cdot \eta)^2]^{n/2} \]
\[ = \log l - \epsilon \log H - (1 + \epsilon) \log H + \frac{1}{2} n \log 1/(1 - (\frac{1}{2} \cdot \eta)^2) + O(\log \log H) \]
\[ = (1 + \epsilon_k) \left[ \frac{1}{1 + \epsilon_k} - (1 + \epsilon) \frac{1}{1 + [\log 1/(1 - (\frac{1}{2} \cdot \eta)^2)]/[2 \log 4/\eta]} - \epsilon \right] \log l \]
\[ + O(\log \log l). \]

Since

\[ \epsilon_k < \frac{\log 1/(1 - (\frac{1}{2} \cdot \eta)^2)}{2 \log 4/\eta}, \]

provided \( \epsilon \) is chosen to be sufficiently small, we have

\[ V_L > [1 - (\frac{1}{2} \cdot \eta)^2]^{n/2}, \]

for all sufficiently large \( l \).

Since

\[ (h - 1)(1 - (\frac{1}{2} \cdot \eta)^2)^{n/2} < 1, \]

for all sufficiently large \( l \), we can now apply the lemma, with \( \gamma = \frac{1}{2} \cdot \eta \), to the set \( S_L \). Thus we can choose a regular \((h - 1)\)-simplex with each of its vertices on \( \Sigma \) within a distance \( \frac{1}{2} \cdot \eta \) of \( S_L \) and with its centre at the centre of \( \Sigma \). So we can choose \( h - 1 \) points \( x_1, x_2, \ldots, x_{h-1} \) of \( L \), each point within a distance \( \frac{1}{2} \cdot \eta \) of a different vertex of a regular \((k - 1)\)-simplex of circum-radius \( 1 \) and edge-length

\[ \sqrt{2(k - 1)/(k - 2)}. \]
Since all the edges of the simplex, $x_1, x_2, \ldots, x_{k-1}$ exceed
\[ \sqrt{\frac{2(k - 1)(k - 2)}{k(k - 1)}} = \sqrt{\frac{2k}{k - 1}}, \]
the subgraph of $G$ with vertices $x_1, x_2, \ldots, x_{k-1}$ is a complete $(k - 1)$-gon, as required. This completes the proof.

References


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