THE MINIMAL REGULAR GRAPH CONTAINING A GIVEN GRAPH

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Let $G$ be an ordinary graph of order $n$ which is not regular and whose maximum degree is $v>0$. Let $H$ denote any regular graph of degree $v$ which contains a subgraph isomorphic to $G$. We seek the minimal order possible for $H$. Let $x_i$ denote the degree of the $i$th vertex in $G$, so $v - x_i$ is the "deficiency" of that vertex; let $\sigma = \Sigma (v - x_i)$ be the sum of the deficiencies and $d$ be the maximum deficiency.

**Theorem.** The necessary and sufficient condition that $m + n$ be the minimal order possible for $H$ is that $m$ be the least positive integer such that:

1. $m \geq \sigma / v$;
2. $m^2 - (v+1)m + \sigma \geq 0$,
3. $m \geq d$ and
4. $(m+n)v$ is an even integer. The maximum value of $m$ is $n$, and for each $n > 3$ there exists a graph $G$ such that $m = n$.

**Proof.** Necessity. It is known that finite graphs $H$ exist, so there is a minimal solution, say a graph $H$ of order $m+n$, and $(m+n)v$ is clearly an even integer.

Let $G'$ be the subgraph of $H$ isomorphic to $G$ and let $A$ be the subgraph induced on the vertices of $H$ not in $G'$. Then in $H$ there are $\sigma$ joins between the subgraphs $G'$ and $A$. Since each of the $m$ vertices of $A$ receives at most $v$ of these $\sigma$ joins, $mv \geq \sigma$, and clearly $m \geq d$.

Denote by $m(A)$ the number of joins in $A$. The sum of the degrees of the vertices of $A$, as points of $A$, must be $mv - \sigma$, hence

(i) \[ m(A) = \frac{1}{2}(mv - \sigma). \]

Then from $m(m-1)/2 \geq m(A)$, it follows that

(ii) \[ m^2 - (v+1)m + \sigma \geq 0, \]

so all four conditions are necessary.

To establish the sufficiency, let $m$ be the least positive integer satisfying conditions (1)-(4). Define a graph $H$ by beginning with $G$ and $m$ extra independent points $a_1, a_2, \ldots, a_m$. Let $p_1, p_2, \ldots, p_k$ denote the points of $G$ with positive deficiencies $d_1, \ldots, d_k$. Let the completion of $G$ be done in the following way. First, $p_1$ is completed by joins to the points $a_1, a_2, \ldots, a_{d_1}$ in succession. Then $p_2$ is completed by joins to successive points $a_i$, starting with $a_{d_1+1}$, which is taken cyclically to be $a_1$ if $d_1 = m$. These completions are possible because $m \geq d$. The degrees attained by points of $A$ in this construction cannot differ from one another at any stage by more than one. So this is also true when the points of $G$ are all complete.

Now let $\sigma/m = h + r/m$, where $h$ and $r$ are nonnegative integers and where $r < m$, and $h < v$ if $r > 0$. Then when the vertices of $G$ have been completed the
set $A$ of vertices $a_i$, $i = 1, \ldots, m$, consists of $r$ points of degree $k+1$ and $m-r$
points of degree $k$. Since there are as yet no joins between points in $A$, any point
of the greatest remaining deficiency $v - k$ can be completed if $v - k \leq m - 1$. But
condition (2) can be written in the form

$$(iii) \quad v - \frac{\sigma}{m} \leq m - 1,$$

from which it follows that

$$(iv) \quad v - k \leq m - 1 + r/m.$$

Because $0 \leq r/m < 1$, while $v - k$ and $m - 1$ are integers, (iv) implies that

$$(v) \quad v - k \leq m - 1.$$

Thus there are in $A$ sufficient points so that each point individually can be com-
pleted.

Finally, the collective completion of all the points in $A$ will be possible if
the sum of the deficiencies is an even integer, that is, if

$$(vi) \quad r(v - k - 1) + (m - r)(v - k) = mv - \sigma$$
is even. But

$$(vii) \quad mv - \sigma = mv - [nv - 2m(G)] = (m + n)v - 2[nv - m(G)].$$
By assumption $(m + n)v$ is even, hence $mv - \sigma$ is even and the completion of all
points in $A$ is possible.

Since $\sigma < nv$, the condition $m \geq \sigma/v$ cannot force $m > n$. Similarly $m^2 - (v + 1)m + \sigma \geq 0$ always holds for $m = v + 1$, and $v + 1 = n$. Condition (3) cannot force $m$ to
exceed $n - 1$. The maximum possible value $m = n$, satisfying conditions (1) and
(2) cannot be increased by condition (4), since $(m + n)v = (n + n)v$ is necessarily
even. Thus in all cases $m \leq n$.

If $n > 3$, let $G$ be the graph obtained from a complete graph of order $n$ by
deleting one join. Then $v = n - 1$ and $\sigma = 2$, and the condition

$$(viii) \quad m^2 - nm + 2 \geq 0$$
implies that $m \geq n$.

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