A Problem Concerning the Zeros of a Certain Kind of Holomorphic Function in the Unit Disk

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To Helmut Hasse on his 65. birthday

Let $f(z)$ be a holomorphic function in the open unit disk D in the complex plane. Suppose that there exists a sequence of distinct Jordan curves $J_1, J_2, \ldots, J_n, \ldots$ in D satisfying the following conditions:

(a) J_n lies in the interior of J_{n+1} $(n = 1, 2, 3, \ldots)$ and

(b) given any $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon)$ such that, for every $n > n_0$, J_n lies in the region $1 - \varepsilon < |z| < 1$.

Set

$$\mu_n = \min_{z \in J_n} |f(z)| \qquad (n = 1, 2, 3, \ldots).$$

If $\lim\limits_{n \to \infty} \mu_n = \infty$, then we call f, for brevity, an *annular function*.

As is evident from this definition, an annular function f is not identically constant, and K, the unit circle, is its natural boundary. Furthermore, according to Kierst and Szpilrajn ([5], p. 291), every holomorphic function in D has at least one asymptotic value, and an annular function evidently can have only ∞ as an asymptotic value; therefore $A(f)$, the set of asymptotic values of f, contains ∞ as its sole element. It is known that annular functions exist; we shall refer to examples later.

If f is an annular function, denote by $Z(f)$ the set of zeros of f. It follows from a theorem of Collingwood and Cartwright ([3], p. 112, Theorem 9, (ii)), that $Z(f)$ is an infinite set of points in D. Let $Z'(f)$ be the set of limit points of $Z(f)$. Then clearly $Z'(f) \subseteq K$. We shall be concerned in this article with the following

Problem: *If f is an annular function, does $Z'(f) = K$?*

It is known that there exist annular functions for which $Z' = K$. A function of Koenigs was shown by Fatou ([4], p. 272) to be of this nature, and annular functions were constructed by Wolff (see [12]) as well as by Bagemihl, Erdös, and Seidel (in [1]) in such a way that $Z' = K$. For each of these functions, every point of K is the end point of an asymptotic path of f.

The following theorem enables us to infer that $Z' = K$ for other known annular functions.

Theorem 1. *Let f be an annular function. Suppose that there exists an everywhere dense subset E of K such that every point of E is the end point of an asymptotic path of f. Then $Z' = K$.*

Proof. First we require some definitions.

Let $\zeta = e^{i\theta}$, and call the extended complex plane Ω. The set $\Gamma(f, |\theta' - \theta| < \eta)$ is defined to be the set of all points $\omega \in \Omega$ with the property that there exists an asymptotic path on which f tends to ω and whose end is contained in the open arc

$$\zeta' = e^{i\theta'}, \ \theta - \eta < \theta' < \theta + \eta.$$

Now put $\chi(f, \zeta) = \bigcap_\eta \Gamma(f, |\theta' - \theta| < \eta)$. Another set that we need is $\Phi(f, \zeta)$, which is defined as the set of all points $\omega \in \Omega$ with the following property. Let $\zeta_1 = e^{i\theta_1}$ and $\zeta_2 = e^{i\theta_2}$ be distinct points of K, with $\theta_1 \leqq \theta \leqq \theta_2$, $0 < \theta_2 - \theta_1 < 2\pi$, and denote by Λ the closed arc $\theta_1 \leqq \arg z \leqq \theta_2$, $|z| = 1$. Suppose that $\{\Lambda_n\}$ is a sequence of Jordan arcs in D, where Λ_n has end points $z_n^{(1)}$ and $z_n^{(2)}$, $\lim_{n\to\infty} z_n^{(1)} = \zeta_1$, $\lim_{n\to\infty} z_n^{(2)} = \zeta_2$, Λ_n is contained in an annulus $1 - \varepsilon_n < |z| < 1$, $\lim_{n\to\infty} \varepsilon_n = 0$, and Λ is the limit of the sequence $\{\Lambda_n\}$. If for every $z \in \Lambda_n$ we have $|f(z) - \omega| < \delta_n$ or $|1/f(z)| < \delta_n$ according as ω is finite or is the point at infinity, where $\lim_{n\to\infty} \delta_n = 0$, then by definition $\omega \in \Phi(f, \zeta)$. Finally, $R(f, \zeta)$ is defined to be the set of values ω such that ω is assumed by f at infinitely many points in every neighborhood of ζ.

Now Collingwood and Cartwright have proved ([3], p. 129, Theorem 16, (ii)) for a meromorphic function f in D, that if $\Gamma(f, |\theta' - \theta| < \eta)$ is of linear measure zero for some $\eta > 0$, then

$$\Omega - R(f, \zeta) \subseteq \chi(f, \zeta) \cup \Phi(f, \zeta). \tag{1}$$

For our annular function f, we have $\Gamma(f, |\theta' - \theta| < \eta) = \{\infty\}$ for every $\eta > 0$, so that this set is of linear measure zero, and hence (1) holds. Clearly $\chi(f, \zeta) = \{\infty\}$, and due to the nature of the set E, we have also that $\Phi(f, \zeta) = \{\infty\}$. It follows from (1) that $0 \in R(f, \zeta)$, and consequently $Z' = K$.

Examples of annular functions in the form of power series $\sum_{k=0}^{\infty} a_k z^{n_k}$ have been given by Lusin and Privalov ([6], p. 148), Davidov (see [10], p. 119), and MacLane ([7], p. 181). An examination of the gaps in these series reveals that in each case

$$\liminf_{k\to\infty} \frac{n_{k+1}}{n_k} > 3. \tag{2}$$

MacLane has shown ([8], p. 46, Theorem 19) that (2) implies the existence of an everywhere dense subset E of K such that every point of E is the end point of an asymptotic path of the function f represented by the series. Hence, according to Theorem 1, $Z'(f) = K$.

Theorem 1 is thus seen to be useful in showing that $Z' = K$ for some annular functions. The next theorem shows, however, that the relation $Z' = K$ is a consequence of a much weaker hypothesis.

If Λ is a path in D terminating in a point $\zeta \in K$, then $C_\Lambda(f, \zeta)$ is defined to be the set of all points $\omega \in \Omega$ with the property that there exists a sequence of points $z_n \in \Lambda$ with $\lim_{n\to\infty} z_n = \zeta$ and $\lim_{n\to\infty} f(z_n) = \omega$.

Theorem 2. *Let f be an annular function. Suppose that there exists an everywhere dense subset E of K such that every point ζ in E is the end point of a path Λ in D with the property that $0 \notin C_\Lambda(f, \zeta)$. Then $Z' = K$.*

Proof. We indicate briefly how this theorem follows from a result established by Ohtsuka ([9], p. 319, Theorem 2).

First of all, it is not difficult to show (see, e. g., [2], p. 1071) that since f is an annular function, the global cluster set of f at any point $\zeta \in K$ (Ohtsuka denotes this set by S_ζ) is Ω.

Take Ohtsuka's w_0 to be the value 0. Then, since $A(f) = \{\infty\}$ for our annular function f, it is easy to see that the set $N_\varrho^{(C^*)}$ in Ohtsuka's theorem is a subset of $\{\infty\}$. Hence, condition (C_1) in that theorem is satisfied, and condition (C_2) is just our assumption concerning the set E localized to a point ζ of K. Since $0 \notin A(f)$, it follows from Ohtsuka's theorem that the value 0 is assumed by f in every neighborhood of ζ, and consequently $Z'(f) = K$.

In view of the fact that for the known analytically defined annular functions it is also known that $Z' = K$, and that there exists a subset E of K of the kind described in Theorem 1, it is perhaps natural to attempt to solve the problem formulated at the beginning of this paper by trying to show that such a set E, or at least a set E of the kind described in Theorem 2, exists for every annular function. This approach is unfruitful, however, because of

Theorem 3. *There exists an annular function f such that $0 \in \Phi(f, \zeta)$ for every $\zeta \in K$.*

Proof. We define f as the product of the following two functions g and h.

Take g to be an infinite product of the sort described in ([1], p. 136). Then g is holomorphic in D, and there exists (see [1], p. 139, Theorem 3) an increasing sequence $\{\varrho_n\}$, $0 < \varrho_n < 1$, $\lim_{n\to\infty} \varrho_n = 1$, such that, setting

$$\mu_n = \min_{|z| = \varrho_n} |f(z)|, \tag{3}$$

we have

$$\lim_{n\to\infty} \mu_n = \infty, \tag{4}$$

so that g is an annular function.

We define the function h by an induction process as an infinite product of polynomials.

First, for every natural number n, let

$$D_n = \{z:\ |z| \leqq \varrho_n\},$$

and put

$$S_n = \left\{z:\ |z| = \frac{2}{3}\varrho_n + \frac{1}{3}\varrho_{n+1},\ -\frac{3\pi}{4} \leqq \arg z \leqq \frac{3\pi}{4}\right\},$$

$$T_n = \left\{z:\ |z| = \frac{1}{3}\varrho_n + \frac{2}{3}\varrho_{n+1},\ \frac{\pi}{4} \leqq \arg z \leqq \frac{7\pi}{4}\right\},$$

$$V_n = S_n \cup T_n.$$

If a function is holomorphic in D_n and continuous on $D_n \cup V_n$, then, as is well known (see [11], p. 47, Theorem 15), it can be approximated arbitrarily closely and uniformly on $D_n \cup V_n$ by a polynomial.

Let $\{\varepsilon_k\}$ be a sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \varepsilon_k = 1. \tag{5}$$

The function that is identically 1 on D_1 and identically 0 on V_1 is holomorphic in D_1 and continuous on $D_1 \cup V_1$. Hence, there exists a polynomial $p_1(z)$ satisfying the conditions

$$| p_1(z) - 1 | < \varepsilon_1 \qquad (z \in D_1),$$

$$| p_1(z) | < \varepsilon_1 \qquad (z \in V_1).$$

Now let $k > 1$, and assume that polynomials $p_j(z)$ and natural numbers $n_j (j = 1, \ldots, k-1)$ have been defined, where $n_1 = 1$.

Since g is an annular function, and $p_1(z) \cdots p_{k-1}(z)$ is bounded in D, there exists a natural number $n_k > n_{k-1}$ such that

$$(6) \qquad | g(z) \cdot p_1(z) \cdots p_{k-1}(z) | > k \qquad (|z| = \varrho_{n_k}).$$

Put

$$(7) \qquad M_k = \max_{z \in V_{n_k}} | g(z) \cdot p_1(z) \cdots p_{k-1}(z) |.$$

As before, there exists a polynomial $p_k(z)$ such that

$$(8) \qquad | p_k(z) - 1 | < \varepsilon_k \qquad (z \in D_{n_k})$$

and

$$(9) \qquad | p_k(z) | < \frac{\varepsilon_k}{M_k} \qquad (z \in V_{n_k}).$$

The sequence of polynomials $\{p_k(z)\}$ is thus defined by induction on k, and we set

$$(10) \qquad h(z) = \prod_{k=1}^{\infty} p_k(z) \qquad (z \in D).$$

Because of (5) and (8), h is holomorphic in D.

Take

$$(11) \qquad f(z) = g(z) \cdot h(z) \qquad (z \in D).$$

Then f is not only holomorphic in D but is also an annular function; for if $|z| = \varrho_{n_k}$, then by (11), (10), (6), and (8) we have

$$\begin{aligned} | f(z) | &= | g(z) \cdot p_1(z) \cdots p_{k-1}(z) | \cdot | p_k(z) p_{k+1}(z) \cdots | \\ &> k \cdot | p_k(z) | \; | p_{k+1}(z) | \cdots \\ &> k \cdot (1 - \varepsilon_k)(1 - \varepsilon_{k+1}) \cdots, \end{aligned}$$

and the last expression tends to ∞ as $k \to \infty$.

If $z \in V_{n_k}$, then using (7), (9), and (8) we find that

$$\begin{aligned} | f(z) | &= | g(z) \cdot p_1(z) \cdots p_{k-1}(z) | \cdot | p_k(z) | \cdot | p_{k+1}(z)\, p_{k+2}(z) \cdots | \\ &\leqq M_k \cdot \frac{\varepsilon_k}{M_k} \cdot | p_{k+1}(z) | \; | p_{k+2}(z) | \cdots \\ &< \varepsilon_k \cdot (1 + \varepsilon_{k+1})(1 + \varepsilon_{k+2}) \cdots, \end{aligned}$$

and the last expression tends to 0 as $k \to \infty$. Bearing in mind the definition of V_{n_k}, we see that this implies that $0 \in \Phi(f, \zeta)$ for every $\zeta \in K$.

The problem formulated in this paper is unsolved, and we hope that what we have said about it will tempt the reader to try to find a solution.

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