Reprinted from JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS. Vol. 15, No 1, July 1966
 Printed in Belgium

On the Solvability of the Equations $[a_i, a_j] = a_r$ and $(a_i', a_j') = a_r'$ in Sequences of Positive Density

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Dedicated to H. S. Vandiver on his eighty-third birthday.

Let $a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Put

$$A(x) = \sum_{a_i < x} 1.$$

We say that the sequence has positive density if $\lim_{x=\infty} A(x)/x$ exists and is positive, the definition of lower (upper) density is self-explanatory. We say that the sequence $a_1 < \cdots$ has positive upper logarithmic density if

$$\limsup_{x=\infty} \frac{1}{\log x} \sum_{a_i < x} \frac{1}{a_i} > 0. \tag{1}$$

Behrend [1] and Erdős [2] proved that if (1) holds then there are infinitely many pairs of a's satisfying $a_i \mid a_j$; and, in fact, Behrend proved that if

$$\sum_{a_i < x} \frac{1}{a_i} > c \frac{\log x}{(\log \log x)^{1/2}} \tag{2}$$

holds for a sufficiently large c and infinitely many x, then $a_i \mid a_j$ has infinitely many solutions. Recently [3], we proved that if $a_1 < \cdots$ is an infinite sequence no term of which divides any other then

$$\sum_{a_i < x} \frac{1}{a_i} = o\left(\frac{\log x}{(\log \log x)^{1/2}}\right). \tag{3}$$

Davenport and Erdős [4] proved that if (1) holds, then, there is an infinite subsequence a_{i_k}, $a_{i_k} \mid a_{i_{k+1}}$, but the following question remained open [5]. Let $a_1 < \cdots$ be a sequence of positive lower density is it true that there are infinitely many triples of distinct a's, a_i, a_j, a_r; a_i', a_j', a_r' satisfying

$$(a_i', a_j') = a_r', \qquad [a_i, a_j] = a_r, \tag{4}$$

where (a_i', a_j') is the greatest common divisor and $[a_i, a_j]$ the least common multiple.

In the present paper we will answer these questions affirmatively. In fact, we shall prove the following stronger ($c, c_1, c_2, \cdots$ denotes suitable positive absolute constants).

THEOREM 1. *Let $a_1 < a_2 < \cdots$ be an infinite sequence of integers for which there are infinitely many integers $n_1 < n_2 < \cdots$ satisfying*

$$\sum_{a_i < n_k} \frac{1}{a_i} > c_1 \frac{\log n_k}{(\log \log n_k)^{1/2}} . \tag{5}$$

Then the equations (4) have infinitely many solutions.

We will easily deduce Theorem 1 from the following combinatorial result of Kleitman [6]: Let S_n be a set of n elements, and A_i, $1 \leqslant i \leqslant r$, $r > c_2 2^n/n^{1/2}$ are subsets of S. Then there are two sets of triples of distinct A's, A_i, A_j, A_r; A_i', A_j', A_r', satisfying

$$A_i \cup A_j = A_r , \qquad A_i' \cap A_j' = A_r'.$$

Before we heard of Kleitman's paper we obtained the same result with $r > c\, 2^n \log \log n / \log n$, this would give instead of (5),

$$c_1 \log n_k \log \log \log \log n_k / \log \log \log n_k .$$

We supress our proof, since it gives a much weaker result and was more complicated than the proof of Kleitman.

Now we deduce Theorem 1 from the result of Kleitman. We only consider the equation $[a_i, a_j] = a_r$, $(a_i', a_j') = a_r'$ can be dealt with similarly. Write

$$a_i = r_i^2 b_i , \qquad b_i \text{ square-free.} \tag{6}$$

The representation (6) is clearly unique. From

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6} < 2$$

and from (5) and (2) it easily follows that there is an r and a subsequence a_{i_j} with $r_{i_j} = r$ and infinitely many values of m for which

$$\sum_{b_{i_j} < m} \frac{1}{b_{i_j}} > \frac{1}{2} c_1 \frac{\log m}{(\log \log m)^{1/2}} . \tag{7}$$

It clearly will suffice to show that (7) implies that

$$[b_u , b_v] = b_w \tag{8}$$

is solvable, since by $a = r^2 b$ (8) implies that (4) holds.

To prove (8) denote by $d_m(k)$ the number of divisors of k among the b's. By (7) we evidently have

$$\sum_{k=1}^{m} d_m(k) = \sum_j \left[\frac{m}{b_{i_j}}\right] \geqslant m \sum_{b_{i_j}<m} \frac{1}{b_{i_j}} - m > \frac{1}{4} c_1 m \frac{\log m}{(\log\log m)^{1/2}}. \tag{9}$$

If $V(k) < \log\log m$ ($V(k)$ is the number of distinct prime factors of k) then, since all the b_{i_j} are square free, we evidently have

$$d_m(k) < 2^{\log\log m}. \tag{10}$$

Thus from (9) and (10) (the dash in the summation indicates that $V(k) \geqslant \log\log m$)

$$\begin{aligned}\sum_{k=1}^{m}{}' d_m(k) &> \frac{1}{4} c_1 m \frac{\log m}{(\log\log m)^{1/2}} - m\, 2^{\log\log m} \\ &> \frac{1}{8} c_1 m \frac{\log m}{(\log\log m)^{1/2}}.\end{aligned} \tag{11}$$

We evidently have

$$\sum_{k=1}^{m} d(k) < 2m \log m.$$

Thus by (11) there clearly exists an integer k satisfying

$$V(k) \geqslant \log\log m \tag{12}$$

and

$$d_m(k) > d(k) \frac{c_1/16}{(\log\log m)^{1/2}}. \tag{13}$$

Without loss of generality we can assume that this k is square-free, since all the b's are square-free. Thus from (12) and (13), we obtain

$$d_m(k) > 2^{V(k)} \cdot \frac{c_1/16}{(\log\log V(k))^{1/2}}. \tag{14}$$

Hence, finally, from (14) with Kleitman's theorem (putting $V(k) = n$, $c_1 > 20c_2$) k has three divisors b_u, b_v, b_w satisfying (8); hence the proof of Theorem 1 is complete.

THEOREM 2. *Let $a_1 < \cdots$ be an infinite sequence of integers for which there are infinitely many integers $n_1 < \cdots$ satisfying*

$$\sum_{a_i<n_k}{}' \frac{1}{a_i} > c_3 \frac{\log n_k}{(\log\log n_k)^{1/4}}.$$

Then there are infinitely many quadruplets of distinct integers a_i , a_j , a_r , a_s *satisfying*

$$(a_i , a_j) = a_r , \qquad [a_i , a_j] = a_s .$$

We supress the proof of Theorem 2, since it is similar to that of Theorem 1; only we here use the following unpublished result of Kleitman: Let $A_i \subset S$, $1 \leqslant i \leqslant r$, $r > c_s 2^n/n^{1/4}$ (S has n elements). Then there are four distinct A's, say A_j , A_j , A_r , A_s , satisfying

$$A_i \cup A_j = A_r , \qquad A_i \cap A_j = A_s .$$

Now we show that Theorem 1 is best possible (except for the value of c_1). In fact, we shall show that there is a sequence $a_1 < a_2 < \cdots$ satisfying for every $x > x_0$

$$\sum_{a_i < x} 1 > \frac{c_5 x}{(\log \log x)^{1/2}} , \tag{15}$$

and such that

$$[a_i , a_j] = a_r \tag{16}$$

is never solvable in distinct integers. (We remind the reader that if (2) holds for infinitely many x, then $a_i \mid a_j$ has infinitely many solutions; but, of course, $[a_i , a_j] = a_r$ is much harder to satisfy than $a_i \mid a_j$.)

We define the sequence of square-free integers $a_1 < a_2 < \cdots$ as follows: Put $\exp z = e^z$. Let $\exp \exp 2k < n < \exp \exp (2k + 1)$, then n is an a if and only if $V(n) = 2k$ and n is odd. If

$$\exp \exp (2k + 1) < n < \exp \exp (2k + 2),$$

then n is an a if and only if $V(n) = 2k$ and n is even. It immediately follows from the results of [7] that our sequence satisfies (15). To complete our proof we show that (16) is not solvable. If $[a_i , a_j] = a_r$ then since

$$V(a_r) > \max (V(a_i), V(a_j)), \tag{17}$$

we have from the definition of the a's:

$$[\log \log a_r] > \max ([\log \log a_i], [\log \log a_j]).$$

On the other hand, from $a_i a_j \leqslant a_r$ and the definition of the a's we have

$$[\log \log a_r] < 2 + \max ([\log \log a_i], [\log \log a_j]).$$

Thus

$$[\log \log a_r] = 1 + \max ([\log \log a_i], [\log \log a_j]). \tag{18}$$

From (17), (18) and the definition of the a's, we obtain by a simple parity consideration that (16) has no solution, which completes our proof.

One would expect that there exists a sequence satisfying (15) for which $(a_i, a_j) = a_r$ is never solvable in distinct integers, but we have not been able to show this.

Perhaps the following result holds: Let $a_1 < \cdots$ be a sequence of positive upper logarithmic density, then there is an infinite subsequence $a_{i_1}, a_{i_2},$ so that the least common multiple of any two a_{i_j}'s is again an a (not necessarily a member of the subsequence a_{i_j}). To show this it would suffice to show that if $a_1 < a_2 < \cdots$ has positive upper logarithmic density, then there is an a_i, so that the set of a_j's for which $[a_i, a_j]$ is again an a has positive upper logarithmic density. We can not decide these questions even if we assume that the a's have positive lower density.

Finally, we remark that for every $\epsilon > 0$ it is easy to construct a sequence of density $> 1 - \epsilon$ for which $a_i \cdot a_j = a_r$ has no solutions, but if the sequence has upper density 1 there always is an infinite subsequence a_{i_j}, $1 \leqslant j < \infty$, so that all the products $\prod a_{i_j}^{\epsilon_j}$, $\epsilon_j = 0$ or 1 are a's (only a finite number of ϵ_j's are 1).

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