SOME REMARKS ON CHROMATIC GRAPHS

BY

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A graph $G$ is said to be $k$-chromatic if its vertices can be split into $k$ classes so that two vertices of the same class are not joined (by an edge) and such a splitting into $k-1$ classes is not possible. The chromatic number will be denoted by $H(G)$. A graph is called complete if any two of its vertices are joined. Denote by $K(G)$ the number of vertices of the largest complete subgraph of $G$. The complementary graph $\bar{G}$ of $G$ is defined as follows: $\bar{G}$ has the same vertices as $G$ and two vertices are joined in $\bar{G}$ if and only if they are not joined in $G$. A set of vertices of $G$ is called independent if no two of them are joined. $I(G)$ denotes the greatest integer for which there is a set of $I(G)$ independent vertices of $G$. We evidently have

$$H(G) \geq K(G) = I(\bar{G}).$$

Throughout this paper $G_n$ will denote a graph of $n$ vertices, $c_1, c_2, \ldots$ will denote positive absolute constants. Vertices of $G$ will be denoted by $X_1, X_2, \ldots$, $G - X_1 - \ldots - X_r$ will denote the graph $G$ from which the vertices $X_1, \ldots, X_r$ and all the edges incident to them have been omitted. $G(X_1, \ldots, X_m)$ denotes the subgraph of $G$ spanned by the vertices $X_1, \ldots, X_m$.

Tutte and Ungar (see [2]) and Zykov [10] were the first to show that for every $l$ there is a graph $G$ with $K(G) = 2$ (i.e. $G$ contains no triangle) and $H(G) = l$. I proved [6] that for every $n$ there is a $G_n$ with $K(G_n) = 2$ and $H(G_n) > cn^{1/2}/\log n$. On the other hand, it easily follows from a result of Szekeres and myself [7] that if $K(G_n) = 2$, then $H(G_n) < c_1 n^{1/2}$.

It is an open and difficult problem to decide if for every $n$ there is $G_n$ with $K(G_n) = 2$ and $H(G_n) > c_2 n^{1/2}$ (P 573).

In the present note we prove the following

THEOREM. For every $n$ there is a $G_n$ satisfying

$$\frac{H(G_n)}{K(G_n)} > \frac{c_3 n}{(\log n)^2}.$$ (1)
But, on the other hand, for every \( G_n \) we have

\[
H(G_n) \leq \frac{1}{K(G_n)} < \frac{c_4 n}{(\log n)^2}.
\]

It seems likely that

\[
\lim_{n \to \infty} \left( \max_{\sigma_n} \frac{H(G_n)}{K(G_n)} \right) / \frac{n}{(\log n)^2} = C.
\]

exists (P 574), but I have not been able to prove (3). By the methods of this note it would be easy to prove that

\[
\frac{(\log 2)^2}{4} \leq \lim \inf_{n \to \infty} \left( \max_{\sigma_n} \frac{H(G_n)}{K(G_n)} \right) / \frac{n}{(\log n)^2}
\]

\[
\leq \lim \sup_{n \to \infty} \left( \max_{\sigma_n} \frac{H(G_n)}{K(G_n)} \right) / \frac{n}{(\log n)^2} \leq (\log 2)^2
\]

First we prove (1). It is known [5] that for every \( n > n_0 \) there is a graph \( G_n \) so that

\[
K(G_n) \leq \frac{2\log n}{\log 2}, \quad K(\overline{G}_n) \leq \frac{2\log n}{\log 2}.
\]

From the definition of the chromatic number we immediately obtain that for every graph \( G_n \)

\[
H(G_n) \geq \frac{n}{I(G_n)} = \frac{n}{K(\overline{G}_n)}.
\]

The proof of (5) is immediate since the vertices of \( G_n \) can be decomposed into \( H(G_n) \) independent sets or \( n \leq H(G_n)I(G_n) \).

(4) and (5) immediately imply (1).

To prove (2) we first prove two simple lemmas.

**Lemma 1.** Let \( \binom{u+v}{v} \geq n \). Then \( uv > c_5(\log n)^2 \).

Without loss of generality we can assume \( u \geq v \). We then have

\[
n \leq \binom{u+v}{v} \leq \binom{2u}{v} \leq \binom{2u}{v} / v! < \left( \frac{2uv}{v} \right)^v.
\]

\[uv > c_5(\log n)^2\] follows from (6) by a simple computation.

In fact, with somewhat more trouble we could prove the following stronger result:

If \( \binom{u+v}{v} \geq n \), then

\[
\min (uv) = \left[ \frac{t}{2} \right] \left[ \frac{t+1}{2} \right],
\]

where \( t \) is the smallest integer for which \( \binom{t}{[t/2]} \geq n \).
From (7) we obtain by a simple computation
\[ uv \geq (1 + o(1)) \left( \frac{\log n}{\log 1} \right)^2. \]

**Lemma 2.** Let \( n \geq m \geq N \). Assume that for every \( m \) and every subgraph \( G(X_1, \ldots, X_m) \) of \( G_n \) we have \( I(G(X_1, \ldots, X_m)) \geq l \). Then
\[ H(G_n) \leq \frac{n}{l} + N. \]

Let \( X_1^{(1)}, \ldots, X_n^{(1)} \) be a maximal system of independent vertices of \( G_n \) \( (n_1 = I(G_n)). \) \( X_1^{(2)}, \ldots, X_{n_2}^{(2)} \) is a maximal system of independent vertices of \( G_n - X_1^{(1)} - \ldots - X_{n_1}^{(1)}; X_1^{(2)}, \ldots, X_{n_3}^{(3)} \) — a maximal system of independent vertices of \( G_n - X_1^{(1)} - \ldots - X_{n_1}^{(1)} - X_1^{(2)} - \ldots - X_{n_2}^{(2)} \) etc. We continue this process until
\[ \sum_{i=1}^r n_i > n - N. \]

By our assumption \( n_i \geq l \) for all \( i, 1 \leq i \leq r \). Thus \( r \leq n/l \). The \( X_1^{(i)}, 1 \leq j \leq n_i, 1 \leq i \leq r \leq n/l \), are the vertices of the \( i \)-th colour and the remaining fewer than \( N \) vertices all get different colours. Thus Lemma 2 is proved.

Now we are ready to prove (2). It is known \([7]\) that
\[ \left( K(G_m) + K(\bar{G}_m) - 2 \right) \geq m. \]

Thus by Lemma 1
\[ K(G_m)K(\bar{G}_m) > c_5(\log m)^2. \]

From (9) we infer that if \( m \geq n/(\log n)^2 \), then for every subgraph \( G(X_1, \ldots, X_m) \) we have
\[ I(G(X_1, \ldots, X_m)) > \frac{c_5(\log m)^2}{K(G(X_1, \ldots, X_m))} > \frac{c_6(\log n)^2}{K(G_n)}. \]

Now apply Lemma 2 with \( N = n/(\log n)^2 \), \( l = c_6(\log n)^2/K(G_n) \). We then obtain
\[ H(G_n) \leq \frac{nK(G_n)}{c_6(\log n)^2} + \frac{n}{(\log n)^2} \]
and (2) immediately follows from (11). This completes the proof of our theorem.

Finally we state some more problems. Denote by \( G(n; m) \) a graph of \( n \) vertices and \( m \) edges. It is easy to see that if \( H(G(n; m)) = k \), then \( m \geq \binom{k}{2} \) and if \( m = \binom{k}{2} \), then \( n = k \), i.e. we have the complete graph
of \(k\) vertices. Determine the smallest integer \(f(l, k)\) for which there exists a graph \(G\) having \(f(l, k)\) edges and satisfying \(K(G) \leq l, H(G) = k\). As we just stated, \(f(k, k) = \binom{k}{2}\) and Dirac showed that \(f(k-1, k) = \binom{k+2}{2} - 5\) (see [3] and [4]). It seems to be very difficult to determine \(f(2, k)\). The graph constructed in [6] shows that \(f(2, k) < c_2 k^3 (\log k)^3\) and it is easy to see that \(f(2, k) > c_3 k^3\). Perhaps

\[
\lim_{k \to \infty} \frac{f(2, k)}{k^3} = C < \infty
\]

exists (P 575).

Denote by \(g(n; l)\) the smallest integer for which there is a \(G(n; g(n; l))\) satisfying \(I(G(n; g(n; l))) = l\). Turán [9] determined \(g(n; l)\) for every \(n\) and \(l\). Let \(g(n; k, l)\) be the smallest integer for which there is a \(G(n; g(n; k, l))\) satisfying

\[
I(G(n; g(n; k, l))) = l, \quad K(G(n; g(n; k, l))) = k.
\]

By (8) we must have \(\binom{k+1-l-2}{k-1} \geq n\). I have not succeeded in determining \(g(n; k, l)\) (P 576).

REFERENCES


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