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ON SOME STATISTICAL PROPERTIES OF THE ALTERNATING GROUP OF DEGREE n

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To the memory of J. Karamata

1. In a sequence of papers [1]) the last two named authors are developing a statistical theory of the symmetric group S_n of n letters. Some of the results can be immediately extended to A_n, to the alternating group of n letters but not all. To mention some, Cauchy has found already that the number of conjugacy-classes of S_n is $p(n)$, the number of unrestricted partitions of n [2]), the same reasoning does not work with A_n. Denoting further the elements of S_n by P, their order by $\mathbf{O}(P)$ and with any fixed real x by $f(n, x)$ the number P's satisfying the inequality

$$\log \mathbf{O}(P) \leqq \tfrac{1}{2}\log^2 n + \frac{x}{\sqrt{3}}\log^{\frac{3}{2}} n \tag{1.1}$$

we proved in III the relation

$$\lim_{n\to\infty} \frac{f(n,x)}{n!} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{Z^2}{2}} dZ\,; \tag{1.2}$$

the corresponding reasoning for A_n must be changed. We proved further in IV that the elements of almost all conjugacy classes in S_n, i.e. with exception of the elements of $o(p(n))$ conjugacy-classes the others can be commuted exactly with

$$\exp\left\{(1+o(1))\frac{\sqrt{6}}{4\pi}\sqrt{n}\log^2 n\right\} \qquad (\exp x = e^x) \tag{1.3}$$

elements of S_n. In what follows we shall prove the following three theorems.

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1) " On some problems of a statistical group theory, I-IV." The first paper is printed in *Zeitschr. f. Wahrscheinlichkeitstheorie und verw. Gebiete*, 4 (1965), pp. 175-186, the second and third in *Acta Math. Acad. Sci. Hung.* T. 18, Fasc. 1-2 (1967), pp. 151-163 resp. T. 18, Fasc. 3-4 (1967), pp. 607-618, the fourth in press. We quote them as I, II, III resp. IV. The sequence will be continued.

2) Throughout this paper two partitions which differ only in the order of summands are considered as identical and the summands are positive integers.

THEOREM I. *The number of the conjugacy-classes* g (n) *in* A_n *is given by*

$$g(n) = \tfrac{1}{2}\, p(n) + \tfrac{3}{2}(-1)^n \sum_{|r| < \sqrt{n}} (-1)^r\, p\left(\frac{n}{2} - \frac{3r^2 + r}{4}\right)$$

$$r \equiv 2n \text{ and } (2n+1) \bmod 4$$

with above defined p (n), *i.e. expressed with the number of conjugacy-classes of* S_n.

Using the classical asymptotical formula of Hardy-Ramanujan [1])

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right) \tag{1.4}$$

which was the subject of several papers of Karamata [2]) it follows at once

$$g(n) = \tfrac{1}{2}\, p(n) + O\left(\frac{1}{\sqrt{n}}\right) \exp\left(\frac{\pi}{\sqrt{3}}\sqrt{n}\right) \sim \frac{1}{8n\sqrt{3}} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right). \tag{1.5}$$

Another, less explicit representation (see (5.7)) will give

$$g(n) - \tfrac{1}{2}\, p(n) > \exp(B\sqrt{n}) \tag{1.6}$$

with an explicit positive numerical B; hence the expectation, $g(n)$ being equal or " very nearly " equal to $\frac{1}{2}\, p(n)$, is false.

Further we shall prove the

THEOREM II. *Denoting for any fixed real* x *by* F (*n*, x) *the number of* P's *in* A_n *satisfying the inequality* (1.1) *the relation*

$$\lim_{n \to \infty} \frac{F(n, x)}{\frac{1}{2}\, n\,!} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\lambda^2}{2}}\, d\lambda$$

holds.

A combination of Theorem I with (1.3) gives at once the

COROLLARY. *For almost all conjugacy-classes of* A_n *(i.e. with exception of* o (g (n)) = o (p (n)) *classes at most) the elements can be commuted exactly with*

$$\exp\left\{(1 + o(1))\frac{\sqrt{6}}{4\pi}\sqrt{n}\log^2 n\right\}$$

elements of A_n.

1) " Asymptotic formulae in combinatory analysis." *Proc. of London Math. Soc.*, 2, XVII (1918) pp. 75-115.

2) See e.g. his paper written with V. AVAKUMOVIC: " Über einige Taubersche Sätze deren Asymptotik von Exponentialcharakter ist, I." *Math. Zeitschr.* 41 (1936), pp. 345-356.

The proofs will illustrate again the role of partitions problems in group theory.

2. For the proof of Theorem I we shall need some lemmata which were partly indicated by Frobenius [1]).

Lemma I. The necessary and sufficient condition that a conjugacy-class H in S_n should be at the same time a conjugacy-class in A_n, is, that it should contain an even permutation P_1 and denoting its centraliser in S_n by $C(P_1)$ this should contain odd permutations too.

Sufficiency. Let P_2 be an arbitrary element of H and

$$P_2 = P_3 P_1 P_3^{-1} \qquad P_3 \in S_n . \tag{2.1}$$

Then (P_2 is even and) P_3 belongs to the coset

$$P_3 \, C(P_1) \tag{2.2}$$

of $C(P_1)$ in S_n. But since $C(P_1)$ contains odd permutations and also even ones (e.g. the unit element) the coset (2.2) contains certainly even permutations too and thus P_2 is conjugate to P_1 in A_n too.

Necessity. Let now H be an arbitrary conjugate class in S_n. The necessity of the existence of an even P_1 in H is evident. If P_4 is an arbitrary odd permutation, the element

$$P_5 = P_4 P_1 P_4^{-1}$$

belongs to H. Since all P 's with

$$P_5 = P P_1 P^{-1}$$

belong to the same coset $P_4 \, C(P_1)$, the fact that $C(P_1)$ contains only even permutations would imply that the whole coset $P_4 \, C(P_1)$ consists of odd permutations, i.e. P_1 and P_5 could not be conjugate in A_n. Q.e.d.

3. Hence the only conjugacy classes of S_n we have to investigate are those with the property the centralisers of all elements consisting of even permutations exclusively. Calling these shortly " bad " classes we assert the

Lemma II. The necessary and sufficient condition for a conjugacy-class H in S_n to be " bad " is that the canonical cycle-representation of its

1) " Uber die Charaktere der alternierenden Gruppe ", Sitzungsberichte der Kön. *Preussischen Akad. d. Wiss. zu Berlin* (1901), pp. 303-315.

elements (which is the same for all as to the number of cycles as well as to their length)

$$(3.1)\quad \begin{cases} a) \text{ should contain no cycles of even length} \\ b) \text{ the occuring odd cycle-lengths are different.} \end{cases}$$

a) *is necessary.* If $P \in H$ and

$$P = (12 \dots 2\nu)(\ )(\ )\dots(\ ).$$

then the permutation

$$\rho = (123 \dots 2\nu)$$

is odd and owing to

$$\rho P \rho^{-1} = P$$

ρ would belong to $C(P)$.

b) *is necessary.* If two cycles of equal length would occur

$$P = (12 \dots \nu)(\nu+1, \dots, 2\nu)(\ )\dots(\ ),$$

then the permutation

$$\rho_1 = (1, \nu+1, 2, \nu+2, \dots, \nu, 2\nu)$$

is odd and owing to

$$\rho_1 P \rho_1^{-1} = P$$

ρ_1 would belong to $C(P)$.

a) *and* b) *are sufficient.* As well-known the order $\mathbf{O}(C(P))$ of the centraliser $C(P)$ of any element of S_n is

$$m_1!\, m_2! \dots m_k!\, n_1^{m_1} n_2^{m_2} \dots n_k^{m_k} \tag{3.2}$$

if the canonical cycle-representation consists of m_ν cycles of length n_ν $(\nu = 1, 2, \dots, k)$, $1 \leqq n_1 < n_2 <, \dots, < n_k$. Thus owing to *a)* and *b)* all m_ν's being 1 we have in our case

$$\mathbf{O}(C(P)) = l_1 l_2 \dots l_k$$

the l_ν's being different odd integers. But then all elements of $C(P)$ are of odd order, i.e. all cycle-lengths are odd and thus all elements of $C(P)$ are even permutations indeed.

4. Hence we characterised all conjugacy-classes of S_n which are not conjugacy-classes in A_n, i.e. which split into more classes. What can be said on their number ?

Lemma III. A conjugacy class in S_n can split only into two conjugacy-classes in A_n at most.

For the proof suppose for a P_1

$$\rho_2 P_1 \rho_2^{-1} = P_6 , \qquad \rho_3 P_1 \rho_3^{-1} = P_7 \tag{4.1}$$

and both can be realised by *odd* ρ_2 and ρ_3 permutations only. Then

$$(\rho_3 \rho_2^{-1}) P_6 (\rho_3 \rho_2^{-1})^{-1} = P_7$$

and $\rho_3 \rho_2^{-1}$ being even, P_6 and P_7 belong to the same conjugacy-class in A_n indeed.

Thus all conjugacy classes of S_n consisting exclusively of even permutations contribute to the total number of conjugacy classes in A_n at least by one; their number is evidently $g_1(n)$ where $g_1(n)$ stands for the number of those partitions of n where the number of summands is congruent to n mod 2. In addition we get owing to lemma II and III one more conjugacy-class in A_n from all conjugacy-classes in S_n which satisfy *a)* and *b)* in (3.1); their number is $g_2(n)$ where $g_2(n)$ stands for the number of those partitions of n consisting of unequal and odd summands. Thus we proved the

Lemma IV. *The total number* g (n) *of conjugacy classes in* A_n *is*

$$g_1(n) + g_2(n) .$$

5. Now we can turn to the proof of Theorem I. Perhaps the shortest way is the following. Let $p_k(n)$ be the number of all partitions of n consisting of k summands. Then we have for $|w| \leq 1$, $|z| < 1$

$$\sum_{k=o}^{\infty} \sum_{m=o}^{\infty} p_k(m)\, w^k z^m = \prod_{\nu=1}^{\infty} \frac{1}{1 - wz^{\nu}} . \tag{5.1}$$

Putting $w = \pm 1$ we get at once

$$\sum_{m=o}^{\infty} z^m \Big(\sum_{\substack{(k) \\ k \text{ even}}} p_k(m) \Big) = \tfrac{1}{2} \Big\{ \prod_{\nu=1}^{\infty} \frac{1}{1 - z^{\nu}} + \prod_{\nu=1}^{\infty} \frac{1}{1 + z^{\nu}} \Big\} \tag{5.2}$$

and

$$\sum_{m=o}^{\infty} z^m \Big(\sum_{\substack{(k) \\ k \text{ odd}}} p_k(m) \Big) = \tfrac{1}{2} \Big\{ \prod_{\nu=1}^{\infty} \frac{1}{1 - z^{\nu}} - \prod_{\nu=1}^{\infty} \frac{1}{1 + z^{\nu}} \Big\} . \tag{5.3}$$

Hence

$$g_1(n) = \tfrac{1}{2} \text{ coeffs } z^n \text{ in } \left\{ \prod_{\nu=1}^{\infty} \frac{1}{1-z^\nu} + (-1)^n \prod_{\nu=1}^{\infty} \frac{1}{1+z^\nu} \right\}$$
$$= \tfrac{1}{2} p(n) + \frac{(-1)^n}{2} \text{ coeffs } z^n \text{ in } \prod_{\nu=1}^{\infty} \frac{1}{1+z^\nu}. \tag{5.4}$$

To get an alternative form of $g_1(n)$ we remark that for $|z| < 1$

$$\prod_{\nu=1}^{\infty} \frac{1}{1+z^\nu} = \prod_{\nu=1}^{\infty} \frac{1-z^\nu}{1-z^{2\nu}} = \prod_{\nu=1}^{\infty} \left(1 - z^{2\nu-1}\right) \tag{5.5}$$

and also

$$\text{coeffs } z^n \text{ in } \prod_{\nu=1}^{\infty} \left(1 - z^{2\nu-1}\right) = (-1)^n \text{ coeffs } z^n \text{ in } \prod_{\nu=1}^{\infty} \left(1 + z^{2\nu-1}\right). \tag{5.6}$$

Thus we get alternatively

$$g_1(n) = \tfrac{1}{2} p(n) + \tfrac{1}{2} \text{ coeffs } z^n \text{ in } \prod_{\nu=1}^{\infty} \left(1 + z^{2\nu-1}\right). \tag{5.7}$$

Owing to Lemma IV we get

$$g(n) - \tfrac{1}{2} p(n) \geqq g_1(n) - \tfrac{1}{2} p(n) =$$
$$= \tfrac{1}{2} \text{ coeffs } z^n \text{ in } \prod_{\nu=1}^{n} \left(1 + z^{2\nu-1}\right). \tag{5.8}$$

Since for real $z \to 1 - 0$

$$\log \prod_{\nu=1}^{\infty} \left(1 + z^{2\nu-1}\right) = \frac{z}{1-z^2} - \tfrac{1}{2} \cdot \frac{z^2}{1-z^4} + \tfrac{1}{3} \cdot \frac{z^3}{1-z^6} - \dots$$
$$\sim \frac{1}{1-z} \left(\frac{z}{1.2} - \frac{z^2}{2.4} + \frac{z^3}{3.6} - \dots \right)$$
$$\sim \frac{1}{1-z} \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \right) \tfrac{1}{2} = \frac{\pi^2}{24} \cdot \frac{1}{1-z}$$

and the assertion (1.6) follows from the Tauberian theorem of Hardy and Ramanujan [1]) at once.

1) " Asymptotic formulae for the distribution of integers of various types." *Proc. of Lond. Math. Soc.* (2), 16 (1917), pp. 117-132.

Returning to Theorem I we need a representation of $g_2(n)$. Since obviously

$$g_2(n) = \text{coeffs} \,.\, z^n \text{ in } \prod_{\nu=1}^{n} (1 + z^{2\nu-1}), \tag{5.9}$$

this, (5.6) and Lemma IV give

$$g(n) = \tfrac{1}{2} p(n) + \tfrac{3}{2} \text{ coeffs} \,.\, z^n \text{ in } \prod_{\nu=1}^{\infty} (1 + z^{2\nu-1}). \tag{5.10}$$

6. In order to get the finite *exact* representation of $g(n)$ given in Theorem I we have to study the representation

$$g(n) = \tfrac{1}{2} p(n) + \tfrac{3}{2} (-1)^n \text{ coeffs} \,.\, z^n \text{ in } \prod_{\nu=1}^{\infty} \frac{1 - z^\nu}{1 - z^{2\nu}},$$

based on (5.5)-(5.6)-(5.10). Then Theorem I follows at once from the identities

$$\prod_{\nu=1}^{\infty} \frac{1}{1 - z^{2\nu}} = \sum_{\nu=o}^{\infty} p(\nu) z^{2\nu}$$

and the classical " Pentagonalzahlsatz " of Euler

$$\prod_{\nu=1}^{\infty} (1 - z^\nu) = \sum_{\nu=-\infty}^{\infty} (-1)^\nu z^{\frac{3\nu^2+\nu}{2}}$$

7. Next we turn to the proof of Theorem II. The proof will be based on the theorem proved in I, according which for almost all $P \in S_n$ $\mathbf{O}(P)$ satisfies the inequality

$$\exp\left(-\log n (\log\log n)^4\right) \leqq \frac{\mathbf{O}(P)}{n_1 n_2 \ldots n_k} \leqq 1; \tag{7.1}$$

here we use again the notation used in **3.** Thus as in III, it will suffice to prove

$$\lim_{n\to\infty} \frac{F^*(n, x)}{\frac{1}{2} n!} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\lambda^2}{2}} d\lambda \tag{7.2}$$

where $F^*(n, x)$ denotes the number of P 's satisfying

$$P \in A_n \tag{7.3}$$

$$n_1 n_2 \ldots n_k \leqq \exp\left(\tfrac{1}{2} \log^2 n + \frac{x}{\sqrt{3}} \log^{\frac{3}{2}} n\right) \tag{7.4}$$

$$1 \leqq n_1 < \ldots < n_k , \qquad k = k(P) . \tag{7.5}$$

If as in **3.** m_ν stands for the number of cycles of length n_ν, we have

$$\sum_{\nu=1}^{k} m_\nu n_\nu = n , \qquad m_\nu \geqq 1 \tag{7.6}$$

the condition (7.3) is equivalent to

$$\sum_{\nu=1}^{k} m_\nu \equiv n \bmod 2 . \tag{7.7}$$

Defining $\sum\limits'_{(7.4)\text{-}(7.5)\text{-}(7.6)\text{-}(7.7)}$ as summation extended to m_ν 's and n_ν 's restricted by (7.4)-(7.5)-(7.6)-(7.7) we define for fixed n

$$\frac{F^*(n, x)}{n\,!} = \sigma_n(x) = \sum{}' \frac{1}{m_1\,! \ldots m_k\,!\; n_1^{m_1} \ldots n_k^{m_k}} \tag{7.8}$$

and

$$\varphi_n(t) = \int_{-\infty}^{\infty} e^{itx}\, d\,\sigma_n(x) \tag{7.9}$$

This gives

$$\varphi_n(t) = \sum_{k=1}^{\infty} \sum_{(7.6)\text{-}(7.7)}{}' \frac{1}{m_1\,!\, m_2\,! \ldots m_k\,!}$$

$$\exp\left\{ t\sqrt{3}\, \frac{\sum\limits_{\nu=1}^{k} \log n_\nu - \frac{1}{2}\log^2 n}{\log^{\frac{3}{2}} n} \right\} \frac{1}{n_1^{m_1} n_2^{m_2} \ldots n_k^{m_k}}$$

or putting

$$\frac{t\sqrt{3}}{\log^{\frac{3}{2}} n} = \tau \tag{7.10}$$

and

$$\varphi_n^*(\tau) = \sum_{k=1}^{\infty} \sum_{(7.6)\text{-}(7.7)} \frac{(n_1 n_2 \ldots n_k)^{i\tau}}{m_1\,!\, m_2\,! \ldots m_k\,!\; n_1^{m_1} n_2^{m_2} \ldots n_k^{m_k}} \tag{7.11}$$

we have

$$\varphi_n(t) = \exp\left(-\frac{it\sqrt{3}}{2} \sqrt{\log n} \right) \varphi_n^*(\tau) . \tag{7.12}$$

Let us define further for integer d

$$\varphi^*_{n,d}(\tau) = \sum_{k=1}^{\infty} \sum\nolimits'' \frac{(n_1 n_2 \dots n_k)^{i\tau}}{m_1 ! \, m_2 ! \dots m_k ! \, n_1^{m_1} \dots n_k^{m_k}}, \tag{7.13}$$

where $\sum''$ is extended to the systems satisfying beside (7.6) also

$$\sum m_\nu \equiv d \bmod 2 . \tag{7.14}$$

8. In order to obtain a more handy representation for $\varphi^*_{n,d}(\tau)$ we fix first the n_ν 's as in (7.5) and consider the infinite series

$$D(z, y, n_1, \dots, n_k) = \sum_{m_\nu \geqq 1} \frac{1}{m_1 ! \, m_2 ! \dots m_k !} \left(\frac{z^{n_1}}{n_1} y \right)^{m_1} \dots \left(\frac{z^{n_k}}{n_k} y \right)^{m_k} . \tag{8.1}$$

This is

$$= \prod_{\nu=1}^{k} \left(-1 + \exp \left(\frac{z^{n_\nu}}{n_\nu} y \right) \right).$$

Putting $y = \pm 1$ we get at once

$$\begin{aligned} \sum_{\substack{m_\nu \geqq 1 \\ \Sigma m_\nu \equiv d \bmod 2}} \frac{1}{m_1 ! \, m_2 ! \dots m_k !} \left(\frac{z^{n_1}}{n_1} \right)^{m_1} \left(\frac{z^{n_2}}{n_2} \right)^{m_2} \dots \left(\frac{z^{n_k}}{n_k} \right)^{m_k} = \\ = \tfrac{1}{2} \left\{ \prod_{\nu=1}^{k} \left(-1 + \exp \frac{z^{n_\nu}}{n_\nu} \right) + (-1)^d \prod_{\nu=1}^{k} \left(-1 + \exp \left(- \frac{z^{n_\nu}}{n_\nu} \right) \right) \right\} . \end{aligned} \tag{8.2}$$

Multiplying by $(n_1 \; n_2 \dots n_k)^{i\tau}$ and performing the summation first for $(n_1, n_2, \dots, n_k)$ in (7.5) and then for k we get from (7.12)

$$\begin{aligned} 1 + \sum_{m=1}^{\infty} \varphi^*_{m,d}(\tau) z^m = \tfrac{1}{2} \left\{ \prod_{l=1}^{\infty} \left(1 + l^{i\tau} \left(e^{\frac{z^l}{l}} - 1 \right) \right) \right. \\ \left. + (-1)^d \prod_{l=1}^{\infty} \left(1 + l^{it} \left(e^{-\frac{z^l}{l}} - 1 \right) \right) \right\} . \end{aligned}$$

Factoring out $\exp \left(\frac{z^l}{l} \right)$ resp. $\exp \left(- \frac{z^l}{l} \right)$ the right side takes the form

$$\begin{aligned} \tfrac{1}{2} \left\{ \frac{1}{1-z} \prod_{l=1}^{\infty} \left(1 + (l^{i\tau} - 1) \left(1 - e^{-\frac{z^l}{l}} \right) \right) \right. \\ \left. + (-1)^d (1-z) \prod_{l=1}^{\infty} \left(1 - (l^{it} - 1) \left(e^{\frac{z^l}{l}} - 1 \right) \right) \right\} \doteq \tfrac{1}{2} \left(\Phi_1(z) + (-1)^d \Phi_2(z) \right) \end{aligned} \tag{8.3}$$

and hence

$$\varphi^*_{n,d}(\tau) = \tfrac{1}{2}\,\text{coeffs}\,.\,z^n \text{ in } \left(\Phi_1(z) + (-1)^d\,\Phi_2(z)\right)$$

and putting $d = n$

$$\varphi^*_n(\tau) = \tfrac{1}{2}\,\text{coeffs}\,.\,z^n \text{ in } \left(\Phi_1(z) + (-1)^n\,\Phi_2(z)\right). \qquad (8.4)$$

Taking in account (7.12) and as proved in III

$$\lim_{n\to\infty} \exp\left(-\frac{it\sqrt{3}}{2}\sqrt{\log n}\right) \text{coeffs}\,.\,z^n \text{ in } \Phi_1(z) = e^{-\frac{t^2}{2}}$$

we get

$$\varphi_n(t) = \tfrac{1}{2}e^{-\frac{t^2}{2}} + O\left(\frac{\log\log n}{\sqrt{\log n}}\right) +$$
$$+ \frac{(-1)^n}{2}\lim_{n\to\infty}\exp\left(-\frac{it\sqrt{3}}{2}\sqrt{\log n}\right)\text{coeffs}\,.\,z^n \text{ in } (1-z)\cdot\prod_{l=1}^{\infty} \qquad (8.5)$$
$$\left\{1 - \left(l^{\frac{it\sqrt{3}}{2}} - 1\right)\left(e^{\frac{z^l}{l}} - 1\right)\right\}.$$

Hence if we can prove that

$$\lim_{n\to\infty}\text{coeffs}\,.\,z^n \text{ in } (1-z)\prod_{l=1}^{\infty}\left\{1 - (l^{it}-1)\left(e^{\frac{z^l}{l}} - 1\right)\right\} = 0\,, \qquad (8.6)$$

the proof of theorem II can be completed as in III.

9. But (8.6) can be proved as follows. Similar process as in III reduces the proof of (8.6) to the proof that for $n \to \infty$

$$\text{coeffs}\,.\,z^n \text{ in } (1-z)\exp\left\{-\frac{it}{\log^{\frac{3}{2}} n}\log^2\frac{1}{1-z} + \right.$$
$$\left. + \frac{t^2}{6\log^3 n}\log^3\frac{1}{1-z}\right\} \to 0\,.$$

holds, which is equivalent to show that for $n \to \infty$

$$\frac{1}{2\pi i}\int_{(L)}\frac{1-z}{z^{n+1}}\exp\left\{-\frac{it}{\log^{\frac{3}{2}} n}\log^2\frac{1}{1-z} + \right.$$
$$\left. + \frac{t^2}{6\log^3 n}\log^3\frac{1}{1-z}\right\}dz \to 0\,. \qquad (9.1)$$

Here as in III L means the following path of integration. Cutting up the $z = x + iy$ -plane along the segment $1 \leqq x < \infty$ L runs along the circle $|z| = 2$ avoiding however the point $z = 1$ by a " Schleife " on both sides of the cut $1 \leqq x \leqq 2$ closing it by the corresponding arc of the circle $|z - 1| = \frac{1}{n}$. The only part of the line-integral in III which did not tend to 0 with $\frac{1}{n}$ was the contribution of the " small " circle; all the corresponding ones tend to 0 also in the present case. The last integral in the present equals to

$$- \frac{1}{n^2} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{2i\varphi}}{\left(1 + \frac{t}{n} e^{i\varphi}\right)^{n+1}} \exp \left\{ - \frac{it}{\log^{\frac{3}{2}} n} (\log n + i(\pi - \varphi))^2 + \frac{t^2}{6 \log^3 n} (\log n + i(\pi - \varphi))^3 \right\} d\varphi \to 0$$

trivially indeed.

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