NON COMPLETE SUMS OF MULTIPLICATIVE FUNCTIONS

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1. It is well known that \( \sum_{d|n} \mu(d) = 0 \) for all \( n > 1 \). We are interested concerning the upper estimate of

\[
M(n) = \max_{z} M(n,z) = \max_{z} \left| \sum_{d|n} \mu(d) \right|
\]

Previously it was proved that

\[
M(n) \leq \left( \frac{\omega(n)}{2} \right)^{2} \leq e^{\frac{\omega(n)}{2}},
\]

where \( \omega(n) \) denotes the number of different prime factors of \( n \) (See [1], [2]).

One of us asked in a recent paper [3] whether \( M(n) \) has a better upper estimate for almost all \( n \). Explicitly it was asked whether

\[
M(n) \leq 2^{\omega(n)}
\]

holds for almost all integers \( n \) with a suitable constant \( x < 1 \). Now we prove a more general theorem, whence (1.2) will immediately follow.

2. Theorem. Let \( f(n) \) be a multiplicative function satisfying the conditions: a) \( |f(n)| < 1 \); b) \( \mathcal{S} \) denote the set of primes \( p \) for which \( f(p) = 1 \), let \( \sum_{p \in \mathcal{S}} \frac{1}{p} = \infty \). Then

\[
\max_{1 \leq z \leq n} \left| \sum_{d|n} f(d) \right| < 2^{\omega(n)}
\]

for almost all \( n \), where \( z \) is an arbitrary constant \( > \frac{1}{2} \).

To prove this we need two lemmas.

Let \( x_1 = \log x, x_2 = \log x_1, y_1 = \log y, y_2 = \log y_1, \Omega(n) \) be the number of all prime divisors of \( n \) counted each of them by their multiplicity. Let \( \epsilon \) be an arbitrarily small positive constant, \( R = (1 - \epsilon)x_2 \). The symbol \( \Sigma' \) denotes a sum extended over those \( n \) for which \( \Omega(n) \leq R \). Since, by the
well known theorem of Hardy and Ramanujan: \( \Omega(n) - x_2 < \varepsilon x_2 \) holds for all \( n \leq x \) except at most \( o(x) \) of them, therefore the \( \sum' \) is extended over almost all \( n \). Let for an arbitrary \( A > 1 \) \( \tau(n', z, A) = \sum_{d|n} \frac{z}{d} < d \leq z \). 

**Lemma 1.** We have 

\[
\left( \sum_{n}^{\text{def}} \right) \sum' \sum_{A^k \leq x} \tau^2(n; A^{k+1}, A) \leq cx \cdot 2^R x_2 (\log A).
\]

**Proof.** For \( d|n \), \( \delta|n \) let \( (d, \delta) = a, d = au, \delta = av, (u, v) = 1 \).

If \( A_k < d < A^{k+1} \), then \( u < n < Au \). Hence

\[
\sum = \sum' \sum_{A^k \leq x} \frac{\tau^2(n; A^{k+1}, A)}{n = au} \leq \sum' \sum_{n = au} 1 \text{ def } \sum_1,
\]

where the last sum is extended over those \( a, u, v, l \) for which \( n \leq x, \Omega(n) \leq \leq R, u \leq v \leq Au \). Therefore \( \sum_1 < \sum u, r \),

\[
\sum_{u, r} = \sum_{m \leq u, r \leq Au} \frac{d(m)}{\omega (m)}.
\]

Since \( d(m) < 2^{\Omega(m)} \), hence

\[
\sum_{u, r} \leq \frac{x}{u v} \cdot 2^{R - \Omega(uv)}
\]

and consequently

\[(2.1) \quad \sum_1 \leq x \cdot 2^R \sum_{u \leq x} x \frac{2 - \Omega(u)}{uv} \leq x \cdot 2^R \sum_{u \leq x} \frac{2 - \Omega(u)}{u} \sum_u,
\]

where

\[(2.2) \quad \sum_u = \sum_{u \leq \tau \leq Au} \frac{2 - \Omega(v)}{v}.
\]

To estimate \( \sum_u \) we use the following theorem due to Hardy and Ramanujan:

if \( \pi_r (y) \) is the number of \( n \leq y \) with \( \Omega(n) = r \), then

\[
\pi_r (y) \leq \frac{y (y_2 + c)^{r-1}}{y_1 (r - 1)!}.
\]

Hence

\[(2.3) \quad \gamma (y) = \sum_{r \leq \tau (y)} 2^{-\pi (r)} \leq \frac{y}{y_1} \sum_{r = 1}^{\tau (y)} \frac{(y_2 + c)^{r-1} 2^{-r+1}}{(r - 1)!} = \frac{y}{y_1} \exp \left[ \frac{1}{2} \frac{y_2 + c}{y_1} \right] < \frac{c}{\sqrt{y_1}}.
\]

Hence

\[
\sum_u \leq \sum_{2 \leq A} \frac{1}{u} \cdot \gamma (u 2^{r+1}) < c \sum_{2 \leq A} \frac{1}{\log u 2^r} \leq c \frac{\log A}{\sqrt{\log u}}.
\]
Taking this estimate into (2.1), we have

\[ \sum_1^{x} \leq c x 2^R \log A \sum_{u \leq x}^{2^{-\alpha(u)}} \frac{1}{\log u}. \]

Furthermore, by (2.3)

\[ \sum_{2 \leq u \leq x}^{2^{-\alpha(u)}} \frac{1}{\log u} \leq c \sum_{2 \leq x}^{1} \frac{1}{\sqrt{t}} \gamma(2t) \leq c \sum_{2 \leq x}^{1} \frac{1}{t} \leq c x_2. \]

Consequently, from (2.4)

\[ \sum_1^{x} \leq c x 2^R (\log A)x_2, \]

which proves the lemma.

Let \( p(n) \) denote the smallest prime divisor \( p \) of \( n \) for which \( p \in \mathcal{S} \), \( p^2 \nmid n \). We take \( p(n) = \infty \) if such \( p \) does not exist.

**Lemma 2.** We have

\[ \lim_{x \to \infty} \frac{1}{x} \sum_{x_0 < x \leq x}^{1} \frac{1}{\log p(n)} = 0, \]

if \( A_x \to \infty \) arbitrarily slowly.

This can be proved, by using the Eratosthenes' sieve; therefore we omit its proof.

3. Now we prove the theorem. We have

\[ \sum_{d \mid n}^{f(d)} = \sum_{d \leq z}^{f(d)} + f(p(n)) \sum_{d \leq z/p(n)}^{f(d)}, \quad n' = \frac{n}{p(n)}. \]

Since \( f(p(n)) = -1 \), therefore

\[ | \sum_{d \mid n}^{f(d)} | \leq 1, \quad \sum_{d \leq z/p(n)}^{f(d)} | \leq \tau(n; z, p(n)). \]

Introducing the notations

\[ C(n) = \max_{d \leq z} | \sum_{d \mid n}^{f(d)} |, \quad T_B(n) = \max_{z \leq x} \tau(n; z, B), \]

from (3.1) we have

\[ C(n) \leq T_B(n) \text{ if } p(n) \leq B. \]

By choosing \( A = B^2 \), we have

\[ T_B^2(n) \leq \max_{k} \tau^2(n; A^k, A) \leq \sum_{A^k \leq x}^{A^k+1} \tau^2(n; A^k, A), \]

and hence, by Lemma 1

\[ \sum^{T_B^2(n)} \leq c x 2^R x_2 \log A. \]
Let $B = x_1$. From (3.3) $T_B(n) \leq 2^{R/2} x_2^2$ for all $n \leq x$ except perhaps some the number of which smaller than $c x x_2^{-2} = o(x)$.

Since

$$p(n) \leq B \quad \text{and} \quad |\Omega(n) - x_2| < c x_2$$

for almost all $n$, therefore, from (3.2)

$$C(n) \leq 2^{R/2} x_2^2 \leq 2^{1+c} x_2^2 \leq 2^{1+c} \frac{x_2}{x} = x_2$$

for almost all $n$.

Using the arbitrariness of $\varepsilon$ we obtain the assertion of the theorem. It is probable that our theorem is nearly best possible. We conjecture that for every $\varepsilon > 0$ and almost all $n$ $M(n) > n^{\frac{1}{2} - \varepsilon}$.

REFERENCES


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