UNSOLVED PROBLEMS IN SET THEORY

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1. **Introduction.** Since 1958 we have published a number of joint papers on set theory [**1**] . . . [**12**] and some triple papers with R. Rado, E. C. Milner, J. Czipszer, G. Födor [**13**] . . . [**17**]. During this period we collected a fair amount of problems we could not solve. Some of them are stated in the papers we have published, some are connected with unpublished results of ours. We were both enthusiastic when we learned that the organizing committee of this symposium was willing to give us an opportunity to publish a paper on these problems.

After having started the work, we immediately realized that the task we have undertaken is not quite as easy and pleasant as we thought it was. First of all, we have problems of very different types.

(A) There are some which seem to be unsolvable, or connected with problems whose independence has already been proved.

(B) There are some which we tried very hard to solve and failed and that is why we feel they are difficult.

(C) There are some which seem to be difficult but we suspect that the difficulty is only technical.

(D) There are some which we only know of, we find them interesting but we simply did not have the time to look at them properly.

(E) There are some which would seem uninteresting to anyone who did not think about them and we would like to publish them all the same, since for one reason or other we are interested in the answer.

On the other hand, there are many cross connections between the different problems and, for lack of time and space, we will be able to give very few of them. In many cases, it will be difficult to tell whose problem it is we are going to state. There are many other problems which arose, but, to make a long story short, we have decided to accept the following principles.

We will restate here quite a lot of the published problems for two reasons. They seem to be more important than some of the unpublished ones, and it is quite difficult to understand the latter without knowing the former

We are going to collect all our problems of type (A), the time and the place being appropriate to put them in the hands of logicians.

We will make comments only on some carefully selected problems, and we will state a lot of others, giving only references (if any) and leave the reader to find out for himself what they are worth. The only help we can give is to indicate to which of the categories (A) . . . (E) we believe the problem stated belongs.

We will try our best to make the references and historical remarks as complete and as fair as possible. If nothing else is stated, we think that the problem in question is due to the two of us, except in §3 where all the problems, if not indicated differently, are due to P. Erdös and R. Rado.

If an open problem depends on several parameters, we usually formulate the instance of it which seems to be the simplest for us.

The order in which the problems are stated does not express any opinion on their importance. We will try to give them in some logical order and to avoid as much new notation as possible.

During our work in set theory, whenever we could not solve a problem, we tried to solve it assuming the generalized continuum hypothesis (G.C.H in what follows). If we still could not solve it, we said that even assuming G.C.H. we do not know the answer. This will be done in this paper too. The word "even" used here is not intended to express any considered opinions or preference. It just describes the way we have been thinking about these problems. If a problem is stated in §§1–7 under the assumption of G.C.H., it means that for various reasons the problem as stated there does not make sense if we do not assume G.C.H.

In the end of our paper we do not bother to state problems without assuming G.C.H., since this would complicate the notations, or the formulation. We think it will be clear for the reader in many cases how to formulate a corresponding problem without assuming G.C.H.

Though many of the difficulties mentioned before seem insurmountable, we still hope that this survey of our problems will be useful.

2. **Notations.** We are going to use the usual classical notations of set theory. It is not appropriate for our present purposes to identify cardinals with initial ordinals. However, finite cardinals and ordinals will not be distinguished.

We point out one difficulty. In many of the different papers on the subject, different kinds of $\rightarrow$ (arrow) relations are introduced and the same notation has been used for different purposes in different papers. Whenever an $\rightarrow$ relation occurs in the text, we will give its definition.

$a, b, c, \ldots, m, n, \ldots$	denote cardinals,
i, j, l	denote integers,
$\alpha, \beta, \gamma, \ldots, \xi, \zeta, \mu, \nu, \rho, \ldots$	denote ordinals.

If a is a cardinal, a^+ is the smallest cardinal greater than a. $\sum$ and $+$ are used to denote both cardinal and ordinal addition. If a is a cardinal, $\Omega(a)$ is its initial number.

By a graph $\mathscr{G} = \langle g, G \rangle$, we mean an ordered pair where g is the set of vertices, G is a set of subsets of two elements of g. The elements of G are the edges of $\mathscr{G}$. For a detailed explanation of the terminology we refer to our paper [10].

3. **Problems formalized with the ordinary partition symbol.** If S is a set, r a cardinal, we put

$$[S]^r = \{X \subset S : |X| = r\}; \qquad [S]^{<r} = \{X \subset S : |X| < r\}.$$

If $[S]^r = \bigcup_{\nu<\varphi} \mathscr{T}_\nu$ the sequence $(\mathscr{T}_\nu)_{\nu<\varphi}$ is said to be an *r-partition of S of type* $|\varphi|$.

Definition of Symbol-I. Let a, b_ν, $\nu < \Omega(c)$ be cardinals or order types, and let c, r be cardinals. Assume further that each b_ν is a cardinal if a is a cardinal. We write $a \to (b_\nu)^r_c$ if the following statement is true.

Let S be a set if a is a cardinal and let S, $\prec$ be a (simply) ordered set if a is an order type, such that $|S| = a$ or typ $S(\prec) = a$ respectively. Let $(\mathscr{T}_\nu)_{\nu<\Omega(c)}$ be an r-partition of type c of S.

Then there exist a $\nu < \Omega(c)$ and a subset $S' \subset S$ such that $[S']^r \subset \mathscr{T}_\nu$ and $|S'| = b_\nu$ or typ $S'(\prec) = b_\nu$ if a is a cardinal or an order type, respectively.

We write $a \nrightarrow (b_\nu)^r_c$ (and in the case of all other symbols to be defined) if this statement is false.

If all the b_ν's equal b, we write $a \to (b)^r_c$. If $c = c_0 + \cdots + c_{n-1}$ and c_i of the b_ν's equal b_i for $i < n < \aleph_0$ we sometimes write

$$a \to ((b_0)_{c_0}, \ldots, (b_{n-1})_{c_{n-1}})^r.$$

If $c_i = 1$, we omit it. Using this terminology, Ramsey's classical theorem [33] can be expressed as follows:

$$\aleph_0 \to (\aleph_0)^r_k \quad \text{if} \quad k, r < \aleph_0.$$

P. Erdös and R. Rado were the first who started to investigate consciously and methodically the possible transfinite generalizations of this theorem, though several other people, e.g. D. Kurepa, have published results which can be expressed using the ordinary partition symbol. A survey of the history of the problem is given in [13]. Erdös and Rado have published with other authors a series of papers on this subject. The symbol in this generality was actually defined in [21]. In their paper [21] they gave a survey of the results and problems known at that time. It is fair to say that their work started all the investigations we are now talking about and though the different problems crystalized by theorems proved by different people it seems to be logical to attribute all the problems concerning Symbol-I (except those involving inaccessible cardinals) to P. Erdös and R. Rado.

3.1. *The ordinary partition symbol in the case of cardinals.* Note that by an early result of P. Erdös and R. Rado, we have

$$a \nrightarrow (\aleph_0, \aleph_0)^{\aleph_0} \quad \text{for every } a;$$

hence we can always assume that $r < \aleph_0$ and the case $r = 1$ is trivial in case of cardinals.

It was realized by P. Erdös and A. Tarski in 1942 [23] that while a direct generalization of Ramsey's theorem fails for cardinals not strongly inaccessible there might be cardinals for which $a \rightarrow (a, a)^2$ holds.

The history of this problem is well known and in this chapter we avoid mention of any problems for Symbol-I in which strongly inaccessible cardinals are involved.

In [13] with Rado we gave a discussion of Symbol-I for cardinals. Using G.C.H. our discussion is almost complete. See main Theorems I and II of [13] on pp. 130 and 138 respectively. The only unsolved problem not involving strongly inaccessible cardinals is highly technical.

Problem 1. *Assume G.C.H.*

$$\aleph_{\omega_{\omega+1}+1} \nrightarrow (\aleph_{\omega_{\omega+1}}, (4)_{\aleph_0})^3?$$

(See [13, Problem 2].)

Recently we have been investigating how far our results and methods cover the problems if we do not assume G.C.H. In case $r = 2$, $b_\nu \geq \aleph_0$ one can obtain a rather complete discussion. These results will be published in detail in a forthcoming book by the three of us. We would like to mention some of the open problems.

In [13] for obtaining negative partition relations our major tool was the negative stepping up lemma.

Problem 2 (Erdös, Hajnal, Rado). *Assume $2 \leq r < \aleph_0$, $a \geq \aleph_0$, b_ν, $(\nu < \Omega(c))$ are cardinals and $a \nrightarrow (b_\nu)_c^r$ holds. Does then $2^a \nrightarrow (b_\nu + 1)_c^{r+1}$ hold? (Here $+$ denotes cardinal additions, i.e., $b_\nu + 1 = b_\nu$ if $b_\nu \geq \aleph_0$.)*

Lemmas 5A, 5F of [13] give this result under different additional assumptions, e.g. if two of the b_ν's are infinite and one is regular. One might guess that this is a problem of type (C) and the answer is affirmative. The most difficult case is when $r = 2$, one b_ν is singular and the others are finite. In this case, we cannot prove the statement even assuming G.C.H. (see Problem 1).

Problem 3 (Erdös, Hajnal, Rado). *Assume that there is an increasing sequence of integers $(n_k)_{k<\omega}$ such that the sequence of cardinals $2^{\aleph_{n_k}}$ is strictly increasing and $2^{\aleph_{n_0}} > \aleph_\omega$. Does then*

$$a = \sum_{k<\omega} 2^{\aleph_{n_k}} \rightarrow (\aleph_\omega, \aleph_\omega)^2?$$

Note if $\aleph_\omega$ is replaced by $\aleph_\alpha$ we have $a \rightarrow \cdots$ or $a \nrightarrow \cdots$ if $\alpha < \omega$ or $\alpha > \omega$, respectively.

Problem 3 might be important, since (without speaking too precisely) it might give an opportunity to show that the truth value of $a \rightarrow (b, c)^2$, a, b, $c \geq \aleph_0$, cannot be computed from the function $\aleph_\alpha^{\aleph_\beta}$.

The only other typical instance in which we cannot tell the truth value of the $\xrightarrow{I}$ in case $r = 2$ with infinite cardinal entries is the following:

Problem 4 (Erdös, Hajnal, Rado). *Put* $\varphi = \Omega((2^{\aleph_0})^+)$ *and assume that the sequence* $\aleph_\alpha^{\aleph_0}$, $\alpha < \varphi$, *is not eventually constant. Put* $a = \sum_{\alpha<\varphi} \aleph_\alpha^{\aleph_0}$. *Does then*

$$a \to (\aleph_\varphi, \aleph_1)^2$$

hold?

(The answer is affirmative if we assume G.C.H. or some other additional assumptions on the function $\aleph_\alpha^{\aleph_\beta}$.)

In case some of the b_ν's are finite many of our results given in **[13]** make use of G.C.H. heavily. It follows easily from the results of **[13]** that

$$\aleph_\omega^{\aleph_0} \nrightarrow (\aleph_{\omega+1}, (\aleph_0)_{\aleph_0})^2$$

and using G.C.H. we proved **[13, Theorem 10]**

$$\aleph_{\omega+1} \nrightarrow (\aleph_{\omega+1}, (3)_{\aleph_0})^2.$$

We cannot fill up the gap between these results if we do not assume G.C.H.

Problem 5 (Erdös, Hajnal, Rado). *Can one prove without assuming G.C.H. that*

$$\aleph_{\omega+1} \nrightarrow (\aleph_{\omega+1}, (3)_{\aleph_0})^2$$

holds?

The general problem of Symbol-I for order types seems to be very ramified. There are only scattered partial results even in case of ordinal numbers.

3.2. *Symbol*-I *in case of denumerable ordinals.* Note that α^β always denotes ordinal power. It seems to be reasonable to consider powers of ω in the first entry. It is easy to see that $\eta \nrightarrow (\omega + 1, 4)^3$ (where η denotes the type of rational numbers); hence we consider only the case $r = 2$. $\omega \to (\omega, \omega)^2$ follows from Ramsey's theorem, $\alpha \nrightarrow (\omega + 1, \omega)^2$ is trivial for every $\alpha < \omega_1$.

E. Specker proved in **[37]**

$$\omega^2 \to (\omega^2, k)^2 \quad \text{for} \quad k < \omega, \tag{1}$$
$$\omega^n \nrightarrow (\omega^n, 3)^2 \quad \text{for} \quad 3 \leq n < \omega.$$

E. C. Milner proved

$$\omega^{\alpha\cdot 3} \nrightarrow (\omega^{\alpha\cdot 2+1}, 3)^2 \quad \text{for} \quad \alpha < \omega_1.$$
$$\omega^4 \to (\omega^3, 3)^2 \tag{2}$$
$$\omega^3 \to (\omega^2 \cdot l, k)^2 \quad \text{for} \quad l, k < \omega.$$

P. Erdös proved

$$\omega^{\alpha\cdot 2+1} \to (\omega^{\alpha+1}, 4)^2 \quad \text{for} \quad \alpha < \omega_1. \tag{3}$$

A. Hajnal proved recently the following theorem. Let $S = \{(n_0, \ldots, n_{k-1})$: $n_i < \omega$ for $i < k\}$ and $[S]^2 = \bigcup_{\nu<l} \mathcal{T}_\nu$ be a 2-partition of type $l < \omega$ of S. Then there exists an infinite set N of integers, for which the partition $(\mathcal{T}_\nu)_{\nu<l}$ is canonical on $S' = \{(n_0, \dots, n_{k-1})\colon\ n_i \in N \text{ for } i < k\}$, i.e. for every pair

$$(n_0, \dots, n_{2k-1})(m_0, \dots, m_{2k-1}),\ n_i, m_j \in N$$
$$n_i < n_j \Leftrightarrow m_i < m_j \quad \text{and} \quad n_i = m_j \Leftrightarrow m_i = m_j$$

implies that $\{(n_0, \dots, n_{k-1})(n_k, \dots, n_{2k-1})\} \in \mathcal{T}_\nu$ holds iff

$$\{(m_0, \dots, m_{k-1})(m_k, \dots, m_{2k-1})\} \in \mathcal{T}_\nu$$

for every $\nu < l$.

ADDED IN PROOF. We learned recently that this result was obtained independently by F. Galvin.

This certainly implies that for every $n < \omega$

$$\omega^n \nrightarrow (\omega^3, f(n))^2 \quad \text{holds for some} \quad f(n) < \omega. \tag{4}$$

If $f(k, n)$ denotes the least integer for which $\omega^n \nrightarrow (\omega^k, f(k, n))^2$ holds for $k \geq 3$, the above mentioned result reduces the determination of $f(k, n)$ to a finite combinatorial problem which is not quite easy to answer. We have computed, e.g. that $f(3, 4) = 5$, but we still do not know whether

Problem 6. $\omega^5 \to (\omega^3, 5)^2$?

However, this is obviously a problem of type (C). The real problem is to determine $f(k, n)$ generally.

None of the results mentioned gives any information about the following problem of type (B):

Problem 7. $\omega^\omega \to (\omega^\omega, 3)^2$?

ADDED IN PROOF (May, 1970). C. C. Chang proved recently $\omega^\omega \to (\omega^\omega, 3)^2$; $\omega^\omega \to (\omega^\omega, 4)^2$ is still open. See C. C. Chang, *A theorem in combinatorial set theory*. Preprint.

3.3. *Symbol*-I *in case of nondenumerable order types and ordinals.* Whenever we have a positive arrow relation $a \to (b_\nu)^r_c$ for cardinals, this obviously implies a corresponding relation for initial ordinals $\alpha \to (\beta_\nu)^r_c$, where α, β_ν are the initial ordinals of a, b_ν, respectively. If $\beta_\nu < \alpha$ for some $\nu < \Omega(c)$ one can ask for what ordinals β'_ν, $|\beta'_\nu| = \beta_\nu$ does the same relation remain true. Usually the method used for the proof of $a \to (b_\nu)^r_c$ yields a slightly stronger result than $\alpha \to (\beta_\nu)^r_c$. There are some results of this type in [21], but most of the problems remain unsolved. By Theorem I of [21], we have $\aleph_1 \to (\aleph_1, \aleph_0)^2$ and the proof in fact gives $\omega_1 \to (\omega_1, \omega + 1)^2$. A. Hajnal proved, using G.C.H. [25] that $\omega_1 \nrightarrow (\omega_1, \omega + 2)^2$ holds.

Problem 8. *Can one prove without using the continuum hypothesis that* $\omega_1 \nrightarrow (\omega_1, \omega + 2)^2$ *holds*?

This might be a problem of type (A).

Using G.C.H. the result of [25] gives $\omega_{\alpha+1} \nrightarrow (\omega_{\alpha+1}, \omega_\alpha + 2)^2$ if $\aleph_\alpha$ is regular and, e.g. Theorem I of [13] implies $\omega_{\alpha+1} \rightarrow (\omega_{\alpha+1}, \omega_{cf(\alpha)} + 1)^2$ for every α. These leave the following problem open.

Problem 9. *Assume G.C.H. Does* $\omega_{\omega+1} \rightarrow (\omega_{\omega+1}, \omega + 2)^2$ *hold*?

On the other hand, G.C.H. $\Rightarrow \omega_{\rho+1} \rightarrow (\omega_\rho + 1, \omega_\rho + 1)^2$ for every ρ.
There is no counterexample for the following:

Problem 10. *Assume G.C.H. Does then* $\omega_{\rho+1} \rightarrow (\xi, \xi)^2$ *hold for every* $\xi < \omega_{\rho+1}$ *and for every* ρ?

By Theorem 8 of [25] we have $\omega_1 \rightarrow (\omega \cdot 2, \omega \cdot k)^2$ for every finite k.
Thus the simplest unsolved instances are

Problem 10/A. *Does*

$$\omega_1 \rightarrow (\omega \cdot 2, \omega^2)^2$$

or

$$\omega_1 \rightarrow (\omega \cdot 3, \omega \cdot 3)^2$$

or

$$\omega_1 \rightarrow (\omega + n, \omega + n, \omega + n)^2 \quad \textit{for every } n < \omega$$

hold?

In cases $\rho > 0$ the problem is even more difficult.

ADDED IN PROOF (May, 1970). F. Galvin communicated to us in a letter that he proved $\omega_1 \rightarrow (\omega \cdot 3, \omega \cdot 3)^2$; $\omega_1 \rightarrow (\omega^2, \omega^2)^2$ is still open.

Problem 10/B. *Assume G.C.H. Does* $\omega^2 \rightarrow (\omega_1 + \omega, \omega_1 + \omega)^2$ *hold*?

10/B might be a problem of type (D) but $\omega_2 \rightarrow (\omega_1 \cdot 2, \omega_1 \cdot 2)^2$ (if true) certainly requires new ideas.

It would be easy to formulate Problem 10 without using G.C.H. We formulate one special case.

Assume $a \geq \aleph_0$. Does then $\Omega((2^a)^+) \rightarrow (\xi, \xi)^2$ hold for every $\xi < \Omega(a^{++})$?
Let λ denote the type of real numbers. There is no counterexample for

Problem 11. *Does* $\lambda \rightarrow (\alpha)^2_k$ *hold for every* $\alpha < \omega_1$, $k < \omega$?

By Theorem 31 of [21] we have $\lambda \rightarrow (\omega + l)^2_3$ for $l < \omega$.

It was proved in [25] that $\lambda \rightarrow (\omega \cdot l, \omega \cdot l)^2$ holds for every $l < \omega$ and $\lambda \rightarrow (\eta, \alpha \text{ or } \alpha^*)^2$ holds for every $\alpha < \omega_1$. The simplest unsolved instances are

Problem 11/A.

$$\lambda \rightarrow (\omega^2, \omega^2)^2?$$
$$\lambda \rightarrow (\omega \cdot 2)^2_3?$$
$$\lambda \rightarrow (\omega + l)^2_4 \quad \textit{for} \quad 2 \leq l < \omega?$$

ADDED IN PROOF. F. Galvin proved $\Phi \rightarrow (\alpha)^2_2$ for every real type Φ and $\alpha < \omega_1$. $\lambda \rightarrow (\alpha)^2_3$ is still unsolved and seems to be difficult.

We mention that all the known results remain valid if λ is replaced by a "real type" Φ (Φ is a real type if $\omega_1, \omega_1^* \not\leq \Phi$). This explains our interest in the following problem of type (E).

Problem 12. *Can one prove a relation $\lambda \to (\Theta_1, \Theta_2)^2$ which does not hold for every real type Φ instead of λ?*

Naturally, one can ask the problems stated in 3.2 for ω_α instead of ω, but there are fewer results.

Specker's result $\omega^3 \not\to (\omega^3, 3)^2$ stated in (1) of 3.2 generalizes easily but the proof of $\omega^2 \to (\omega^2, k)^2$ uses finitely additive measures and breaks down for ω_1.

In a forthcoming paper [22] P. Erdös and R. Rado state

(1) for every ξ and for every finite k, l there exists an $n < \omega$ such that $\omega_\xi \cdot n \to (\omega_\xi \cdot k, l)^2$,

and A. Hajnal proved

(2) assume G.C.H., then $\omega_{\xi+1} \cdot \rho \not\to (\omega_{\xi+1} \cdot \omega_\xi, 3)^2$ for every $\rho < \omega_{\xi+1}$ and for every ξ.

The following remained unsolved:

Problem 13. *Assume G.C.H.*

$$\omega_1^2 \to (\omega_1^2, 3)^2?$$
$$\omega_2 \cdot \omega \to (\omega_2 \omega, 3)^2?$$

The first one seems to be of type (B). Note that $\omega_\alpha^2 \not\to (\omega_\alpha + 1, \omega)^2$ is known for every α.

ADDED IN PROOF (May, 1970). A. Hajnal proved G.C.H. $\Rightarrow \omega_1^2 \not\to (\omega_1^2, 3)^2$. See A. Hajnal, *A negative partition relation*, Proc. Nat. Acad. Sci. U.S.A. (to appear).

4. **Problems for Symbol-II.**

DEFINITION OF SYMBOL-II. Let a, b_ν, r, c have the same meaning as in the definition of Symbol-I.

We write $a \xrightarrow{\text{II}} [b_\nu]_c^r$ if the following statement is true.

Let S be a set if a is a cardinal and S, $\prec$ be a (simply) ordered set if a is an order type such that $|S| = a$ and type $S(\prec) = a$ respectively. Let further $(\mathscr{T}_\nu)_{\nu < \Omega(c)}$ be an r-partition of type c of S. Then there are a subset $S' \subset S$ and an ordinal $\nu_0 < \Omega(c)$ such that $[S']^r \subset \bigcup_{\nu \neq \nu_0 < \Omega(c)} \mathscr{T}_\nu$ and $|S'| = b_{\nu_0}$ or typ $S'(\prec) = b_{\nu_0}$ respectively.

We will use the same self-explanatory abbreviations in cases when some of the b_ν's are equal which were introduced for Symbol-I.

Symbol-II was first defined for cardinals in [13] and was not yet so thoroughly investigated as Symbol-I. It is obvious that, e.g. $a \not\to [b]_c^r$, $c > 2$ is a much stronger counterexample than $a \not\to (b, b)^r \Leftrightarrow a \not\to [b, b]^r$.

We mention that in [9] we have proved that if $a \not\to [a]_a^r$ for some $r < \omega$ then there is a Jónsson algebra of power a.

4.1. *The case of the infinite exponents.* One would expect that as in case of Symbol-I we have a best possible negative result for $r \geq \aleph_0$. There is no counter-example for the following.

Problem 14. *Assume $a, b \geq r \geq \aleph_0$ are cardinals. Then $a \nrightarrow [b]^r_{b^r}$.*

Note that $b \rightarrow [b]^r_c$ is trival for $c > b^r$. Using an idea of J. Novák described in [1] we can prove the following result.

THEOREM (UNPUBLISHED). *Assume $a, r \geq \aleph_0$ are cardinals. Then $a \nrightarrow [r]^r_{2^r}$.*

The following simple instance of Problem 14 remains unsolved.

Problem 14/A. *Is it true that*

$$a \nrightarrow [(2^{\aleph_0})^+]^{\aleph_0}_{(2^{\aleph_0})^+}$$

holds for every a? Or assuming G.C.H. is it true that $a \nrightarrow [\aleph_2]^{\aleph_0}_{\aleph_2}$ holds for every a?

Note that for $a < \aleph_\omega$ this might be a consequence of $\overset{\mathrm{II}}{\nrightarrow}$ relations with finite exponents. See Problem 17/A, and that assuming G.C.H. we can in fact prove the $\overset{\mathrm{II}}{\nrightarrow}$ relation for the special case $a = \aleph_2$.

4.2. *Symbol*-II, *in case $r < \omega$.* Theorem 17 of [13] states that G.C.H. $\Rightarrow$ $\aleph_{\alpha+1} \nrightarrow [\aleph_{\alpha+1}]^2_{\aleph_{\alpha+1}}$ for every α and it is well known that $2^{\aleph_\alpha} \nrightarrow (\aleph_{\alpha+1}, \aleph_{\alpha+1})^2$ i.e. $2^{\aleph_\alpha} \nrightarrow [\aleph_{\alpha+1}]^2_2$.

Problem 15. *Can one prove without assuming C.H.*

$$2^{\aleph_0} \nrightarrow [\aleph_1]^2_3 \quad or \quad 2^{\aleph_0} \nrightarrow [2^{\aleph_0}]^2_3 \quad or \quad \aleph_1 \nrightarrow [\aleph_1]^2_3?$$

This might be a problem of type (A).

ADDED IN PROOF (May, 1970). We learned from a letter of F. Galvin that he proved $2^{\aleph_0} \rightarrow [2^{\aleph_0}]^2_n$ for $n < \omega$ and $\aleph_1 \nrightarrow [\aleph_1]^2_4$.

We would like to stress the importance of the following:

Problem 16. *Let a be an inaccessible cardinal for which $a \nrightarrow (a, a)^2$ holds. Does then $a \nrightarrow [a]^2_a$ hold?*

We do not know if $a \nrightarrow [a]^2_a$ holds for the first strongly inaccessible cardinal $a > \aleph_0$.

We think that the problem whether $a \nrightarrow [a]^2_a$ holds is strongly connected with the following. There exists an a-complete field S of subsets of a set X of power a generated by at most a elements $[X]^{<a} \subseteq S$ in which there is no a-complete proper a-saturated ideal I such that $[X]^{<a} \subset I$.

As to further details of the results concerning $a \rightarrow [b_\nu]^r_c$ a, b_ν cardinals we refer to [13]. We formulate only one more problem.

Problem 17 (Erdös, Hajnal, Rado). *Assume $a \geq \aleph_0$, $2 \leq r < \aleph_0$, $b_\nu > r$ are cardinals and $a \nrightarrow [b_\nu]^r_c$. Does then $2^a \nrightarrow [b_\nu + 1]^r_c$ hold?*

We cannot give a positive answer even assuming G.C.H.

Problem 17/A. *Does* $2^{2^{\aleph_0}} \nrightarrow [\aleph_1]^3_{\aleph_1}$ *hold? Or does G.C.H.* $\Rightarrow \aleph_2 \nrightarrow [\aleph_1]^3_{\aleph_1}$ *and* $\aleph_{l+1} \nrightarrow [\aleph_1]^{l+2}_{\aleph_1}$ *for* $l < \omega$?

This should be compared with Theorems 17, 25, and Problem 3 of **[13]**. As far as we know no one has investigated Symbol-II for types. There are some very simple problems we cannot answer. Here is one of them.

Problem 18. $\omega^\omega \rightarrow [\omega^\omega]^2_{\aleph_0}$?

Note that $\omega^k \rightarrow [\omega^k]^2_{\aleph_0}$, $k < \omega$ follows from the theorem of A. Hajnal mentioned in subsection 3.2.

5. **Symbol-III and related problems.**

Definition of Symbol-III. Let r, c, d be cardinals, a, b types or cardinals but b should be a cardinal if a is a cardinal.

$a \xrightarrow{\text{III}} [b]^r_{c,d}$ denotes that the following statement is true.

Let S be a set if a is a cardinal and let S, $\prec$ be an ordered set if a is a type such that $|S| = a$ or typ $S(\prec) = a$ respectively. Let further $(\mathscr{T}_\nu)_{\nu<\Omega(c)}$ be an r-partition of type c of S. Then there exist an $S' \subseteq S$ and a set N of ordinals less than $\Omega(c)$ such that $|N| \leq d$, $[S']^r \subseteq \bigcup_{\nu\in N} \mathscr{T}_\nu$ and $|S'| = b$ or typ $S'(\prec) = b$ respectively.

Symbol-III is Symbol-V of **[13]** defined in **[13**, 18.3**]**. We collected a number of results and problems in **[13**, 18**]** which we do not repeat here; we only point out one of them (Problem 3.1(a) of **[13]**).

Problem 19. *Assume G.C.H. Does then*

$$\aleph_2 \overset{\text{III}}{\nrightarrow} [\aleph_1]^2_{\aleph_1,\aleph_0}$$

hold?

We came to this problem when considering a problem of S. Ulam. Several other people have independently considered this problem and though we were unable to collect references we know by hearsay that both $\aleph_2 \rightarrow [\aleph_1]^2_{\aleph_1,\aleph_0}$ and $\aleph_2 \nrightarrow [\aleph_1]^2_{\aleph_1,\aleph_0}$ are proved to be consistent with the axioms of set theory and G.C.H. E.g. F. Rowbottom proved that the negative relation follows from Gödel's axiom $V = L$.

We formulated this problem because we will formulate a series of other problems related to it and some implications between them.

Let $(A_\nu)_{\nu<\varphi}$ be a sequence of disjoint sets. The set X is said to be a transversal of the sequence $(A_\nu)_{\nu<\varphi}$ if $|A_\nu \cap X| = 1$ for every $\nu < \varphi$.

Problem 19/A. *Assume G.C.H. Let* $(A_\nu)_{\nu<\omega_1}$ *be a sequence of disjoint sets such that* $|A_\nu| = \aleph_0$ *for every* $\nu < \omega_1$.

Does there exist a system $\mathscr{F}$, $|\mathscr{F}| = \aleph_2$ *of almost disjoint transversals* (*i.e.* $X \neq Y \in \mathscr{F}$ *implies* $|X \cap Y| \leq \aleph_0$)?

Problem 19/B. *Does there exist under the conditions of Problem* 19/A, *a system* $\mathscr{F}$, $|\mathscr{F}| = \aleph_2$ *of almost disjoint transversals satisfying the following additional* *condition? If $X, Y \in \mathscr{F}$ and $X \cap A_\nu = Y \cap A_\nu$ for some $\nu < \omega_1$ then $X \cap A_\mu = Y \cap A_\mu$ for every $\mu < \nu$.*

It is easy to see that Problem 19/B is equivalent to the well-known Kurepa problem. It is also quite easy to see that Problem 19/B $\Rightarrow$ Problem 19/A $\Rightarrow$ Problem 19. We cannot answer the following problem.

Problem 19/C. *Does either of the implications*

$$\text{Problem 19} \Rightarrow \text{Problem 19/A}, \qquad \text{Problem 19/A} \Rightarrow \text{Problem 19/B}$$

hold?

All these problems are well known.

Let a be a regular cardinal, $a > \aleph_0$. Let X be a set of ordinals less than $\Omega(a)$. A function f defined on X with ordinal values less than $\Omega(a)$ is said to be *regressive* on X if $f(\xi) < \xi$ for every $\xi \in X$. X is said to be stationary (in $\Omega(a)$) if for every regressive function f on X there is a $\rho < \Omega(a)$ such that $(f^{-1}(\rho)) = a$.

It has been recently proved by R. Solovay that every stationary X is the union of a disjoint stationary sets.

Problem 19/D. *Does there exist a system $\mathscr{F}$, $|\mathscr{F}| = \aleph_2$ or $2^{\aleph_1}$ of almost disjoint stationary subsets of ω_1?*

We do not know the answer for any regular a in place of ω_1. It is easy to see that a positive answer to Problem 19/A implies a positive answer to Problem 19/D.

Some problems related to 19/A will be considered in a forthcoming paper by E. C. Milner and the two of us [17].

We mention one more problem of Kurepa type. We do not know if its independence has already been investigated.

Problem 19/E. *Assume G.C.H. Let $|S| = \aleph_\omega$. Does there exist a family $\mathscr{F}$, $|\mathscr{F}| = \aleph_{\omega+1}$, $\mathscr{F} \subseteq [S]^{\aleph_0}$ such that $\mathscr{F} \restriction X = \{F \cap X : F \in \mathscr{F}\}$ has power $\aleph_0$ for every $X \subset S$, $|X| = \aleph_0$?*

We turn to a problem concerning Symbol-III in case of ordinals. We mention that even Symbol-I yields interesting problems for ordinals in case $r = 1$, but these had been solved and completely discussed by E. C. Milner and R. Rado in [32]. One of their surprising results states that $\rho \not\rightarrow (\omega_\xi^\omega)^1_{\aleph_0}$ holds for every $\rho < \omega_{\xi+1}$ and for every ξ.

This can be formulated in terms of Symbol-III: $\rho \not\rightarrow [\omega_\xi^\omega]^1_{\aleph_0, <\aleph_0}$ holds for every $\rho < \omega_{\xi+1}$ and for every ξ.

(Symbol-III was not defined with $<\aleph_0$ in place of d, but has a self-explanatory meaning.)

A straightforward generalization of this would be the following:

Problem 20. *Let $\rho < \omega_{\xi+1}$, $\zeta + 1 < \xi$. Does then*

$$\rho \not\rightarrow [\omega_\xi^{\omega\zeta+1}]^1_{\aleph_{\zeta+1}, \aleph_\zeta}$$

hold?

This is certainly a problem of type (A) and we would be interested if a positive answer for it is consistent with the axioms of set theory.

We wish to make some remarks on a special case of it.

Problem. 20/A. *Let $\rho < \omega_3$. Does then $\rho \nrightarrow [\omega_2^{\omega_1}]^1_{\aleph_1,\aleph_0}$ hold for every $\rho < \omega_3$?*

We know the following partial results:

(1) $\rho \nrightarrow [\omega_2^{\omega_1}]^1_{\aleph_1,\aleph_0}$ for $\rho < \omega_2^{\omega_2}$.

(2) If $\omega_2^{\omega_2} \nrightarrow [\omega_2^{\omega_1}]^1_{\aleph_1,\aleph_0}$, then the answer for Problem 19/D is affirmative. We formulate two more problems of this type.

Problem 21. *Let $\rho < \omega_3$. Does there exist a sequence $(f_\nu^\rho)_{\nu<\rho}$ of type ρ of functions defined on ordinals $< \omega_1$ with values $< \omega_1$ satisfying the following condition? Whenever $\nu_1 < \nu_2 < \rho$ then the set*

(0) $\{\xi < \omega_1 : f^\rho_{\nu_1}(\xi) \geq f^\rho_{\nu_2}(\xi)\}$ is nonstationary in ω_1.

Problem 21/A. *Does there exist a sequence of functions satisfying the requirements of Problem 21 and the stronger conditions?*

If $\nu_1 < \nu_2 < \rho$ then

(00) $|\{\xi < \omega_1 : f^\rho_{\nu_1}(\xi) \geq f^\rho_{\nu_2}(\xi)\}| < \aleph_1$.

We can prove

(3) A positive answer to Problem 20/A implies that the answer is positive for 21.

(1), (2), (3) will be published in a triple paper with Milner. It is obvious that Problem 21/A $\Rightarrow$ Problem 21.

It is also obvious that a positive answer for Problem 21 and Problem 21/A in the special cases $\rho = \omega_2 + 1$ implies a positive answer to Problem 19/D, Problem 19/A respectively. On the other hand, we do not know if the consistency of Problem 21 or Problem 21/A has already been investigated.

6. Symbol-IV, $a \to (b)_c^{<\aleph_0}$ and related problems.

DEFINITION OF SYMBOL-IV. Let a, b, c be cardinals. $a \to (b)_c^{<\aleph_0}$ denotes that the following statement is true. Let S be a set $|S| = a$. Let $(\mathscr{T}_\nu^r)_{\nu<\Omega(c)}$ be an r-partition of type c of S for every $r < \omega$. Then there exist an $r_0 < \omega$, a function $\nu(r) < \Omega(c)$ for $r < \omega$ and a subset $S' \subset S$, $|S'| = b$ such that $[S']^r \subset \mathscr{T}^r_{\nu(r)}$ for every $r_0 \leq r < \omega$.

Symbol-IV is Symbol-II of [13]. We have proved in [1] that $a \to (a)_c^{<\aleph_0}$ holds for $c < a$ if a is a measurable cardinal $> \aleph_0$. J. Silver has proved recently [34] that $a \nrightarrow (\aleph_0)_2^{<\aleph_0}$ holds for a very large section of cardinals. For other results, history, and references see [13] and [34].

We will speak about some strongly related problems.

DEFINITION OF SYMBOL-IV.1. Let a and b be cardinals. $a \Rightarrow (b)^{<\aleph_0}$ denotes that the following statement is true. Let S be a set, $|S| = a$, and let $(\mathscr{T}_0^r, \mathscr{T}_1^r)$ be an r-partition of type 2 of S for every $r < \omega$, such that

(0) $X \subset S$, $|X| = r + 1$ implies $[X]^r \not\subset \mathscr{T}_0$ for every $r < \omega$. Then there exist an $r_0 < \omega$ and $S' \subset S$ such that $|S'| = b$ and $[S']^r \subset \mathscr{T}_1^r$ for every $r_0 \leq r < \omega$.

It is obvious that $a \not\Rightarrow (b)^{<\aleph_0}$ implies $a \nrightarrow (b)_2^{<\aleph_0}$.

We have proved several years ago that all the negative results stated for Symbol-IV (i.e. Symbol-II with $c = 2$ in [13]) are valid for Symbol-IV.1 as well.

The following problem (of type (E)) arises.

Problem 22. *Does $a \not\Rightarrow (\aleph_0)^{<\aleph_0}$ hold for the first strongly inaccessible cardinal $a > \aleph_0$ (or for a large section of cardinals)?*

We can prove that if we had defined a symbol $a \Rightarrow_l (b)^{<\aleph_0}$ by replacing the condition (0) of the definition of Symbol-IV.1 by the stronger condition

(00) $X \subset S$, $|X| = r + 1$ implies $|[X]^r \cap \mathscr{T}_0| < r + 1 - l$ for $r < \omega$ we would have had $2^{\aleph_0} \Rightarrow_l (\aleph_0)^{<\aleph_0}$ for some $l < \omega$, but we do not know if l can be replaced by 1.

We define a Symbol-IV.2 which is in the same relation to Symbol-IV as Symbol-II is to Symbol-I.

Definition of Symbol-IV.2. Let a, b, c be cardinals.

$a \to [b]_c^{<\aleph_0}$ is said to hold if the following statement is true.

Let S be a set $|S| = a$. Let further $(\mathscr{T}_\nu^r)_{\nu<\Omega(c)}$ be an r-partition of type c of S for every $r < \omega$. Then there exist $S' \subset S$, $r_0 < \omega$ and a function $\nu(r) < \omega$ for $r < \omega$ such that

$$[S_1']^r \subset \bigcup_{\nu \neq \nu(r);\, \nu<\Omega(c)} \mathscr{T}_\nu^r \quad \text{for every} \quad r < \omega.$$

It is obvious that $a \nrightarrow [b]_c^{<\aleph_0}$ for $c \geq 2$ is a stronger counterexample than $a \nrightarrow (b)_2^{<\aleph_0}$.

We cannot even decide

Problem 23. $\aleph_0 \overset{\text{IV.2}}{\nrightarrow} [\aleph_0]_{\aleph_0}^{<\aleph_0}$?

We always suspected that there is a $\nrightarrow$ relation. We do not know if $a \to [\aleph_0]_{\aleph_0}^{<\aleph_0}$ holds for any "relatively small" cardinal.

This might be a problem of type (D).

Added in proof. Let $a \to (b)^{<\aleph_0}_{c_0,\dots,c_r,\dots,d_0,\dots,d_n,\dots}$ denote the following statement:

Let S be a set, $|S| = a$. Let further (J_ν^r), $\nu < \Omega(c_r)$ be an r partition of type c_r of S for every $r < \omega$.

Then there are sets B_r of ordinals less than $\Omega(c_r)$ and $S' \subset S$ such that

$$[S]^r \subset \bigcup_{\nu\in B_r} J_\nu^r, \qquad |S'| = b \text{ and } |B_r| \leq d_r \text{ for } r < \omega.$$

J. E. Baumgartner and independently R. Rado and we proved that $\aleph_0 \to [\aleph_0]^{<\aleph_0}_{c_0,\dots,c_r,\dots,d_0,\dots,d_r,\dots}$ holds provided $c_r < \omega$ and $d_r \to +\infty$ if $r \to +\infty$. This implies obviously a negative solution of Problem 23.

7. Polarized partition relations and related problems.

Definition of Symbol-V (see [13, 3.3]). (1) Let $a, b, c, d_\nu, e_\nu, f_\nu, g_\nu, \nu < \Omega(c)$ be cardinals.

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} d_\nu & & f_\nu \\ & \vee & \\ e_\nu & & g_\nu \end{pmatrix}_c$$

is said to hold if the following statement is true.

Whenever S_0, S_1 are sets such that $|S_0| = a$, $|S_1| = b$ and $(\mathscr{T}_\nu)_{\nu<\Omega(c)}$ is a partition of $S_0 \times S_1$ then there exist $\nu_0 < \Omega(c)$, $S_0' \subset S_0$, $S_1' \subset S_1$ such that $S_0' \times S_1' \subset \mathscr{T}_{\nu_0}$ and either $|S_0'| = d_\nu$ and $|S_1'| = e_\nu$ or $|S_0'| = f_\nu$ and $|S_1'| = g_\nu$.

We write

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} d_\nu \\ e_\nu \end{pmatrix}_c$$

if $d_\nu = f_\nu$, $e_\nu = g_\nu$ for $\nu < \Omega(c)$.

(2) Let $r = r_0 + \cdots + r_{l-1}$, $l < \omega$ for an $r < \omega$.

Let a_i, $b_{i,\nu}$, c be cardinals for $i < l$, $\nu < \Omega(c)$.

$$\begin{pmatrix} a_0 \\ \cdot \\ \cdot \\ \cdot \\ a_{l-1} \end{pmatrix} \rightarrow \begin{pmatrix} b_{0,\nu} \\ \cdot \\ \cdot \\ \cdot \\ b_{l-1,\nu} \end{pmatrix}^{r_0,\ldots,r_{l-1}}$$

is said to hold if the following statement is true.

Whenever S_i, $i < l$ are sets such that $|S_i| = a_i$ for $i < l$ and $(\mathscr{T}_\nu)_{\nu<\Omega(c)}$ is a partition of $[S_0]^{r_0} \times \cdots \times [S_{l-1}]^{r_{l-1}}$ then there are subsets $S_i' \subset S_i$ for $i < l$ and $\nu_0 < \Omega(c)$ such that $|S_i'| = b_{i,\nu_0}$ for $i < l$ and

$$[S_0']^{r_0} \times \cdots \times [S_{l-1}']^{r_{l-1}} \subset \mathscr{T}_{\nu_0}.$$

Obviously

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \rightarrow \begin{pmatrix} b_{0\nu} \\ b_{1\nu} \end{pmatrix}_c^{1,1}$$

means the same as

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \rightarrow \begin{pmatrix} b_{0\nu} \\ b_{1\nu} \end{pmatrix}.$$

One could give a definition of Symbol-V for types under obvious restrictions for the entries as in the case of the previous symbols. It is also obvious that as in case of Symbols-I,-II a corresponding "square bracket" symbol can be defined, e.g. in the definition of

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} b_{0\nu} \\ b_{1\nu} \end{pmatrix}_c$$

$S_0' \times S_1' \subset \mathscr{T}_{\nu_0}$ has to be replaced by

$$S_0' \times S_1' \subset \bigcup_{\nu_0 \neq \nu;\, \nu<\Omega(c)} \mathscr{T}_\nu.$$

In **[13]** we have investigated the symbol as defined in (1) with $c = 2$, and assuming G.C.H. in almost every case.

We mention the following

Problem 24 (Erdös, Hajnal, Rado). *Assume G.C.H. Does then*

$$\binom{\aleph_2}{\aleph_1} \to \binom{\aleph_i}{\aleph_1}, \binom{\aleph_j}{\aleph_1}$$

hold for $0 \leq i, j \leq 1$?

Problem 25 (Erdös, Hajnal, Rado). *Assume G.C.H.*

$$\binom{\aleph_{\omega+1}}{\aleph_{\omega+1}} \to \binom{\aleph_{\omega+1}}{\aleph_\omega} \wedge \binom{\aleph_1}{\aleph_\omega}?$$

See Problems 12 and 14 of **[13]**. For a discussion of the known results see **[13]**.

We think that both problems might be of type (A).

We mention that unlike in the case of Symbol-I most of our results use G.C.H. essentially and even the simplest problems seem to be unsolvable.

We do not know the answer to

Problem 26. *Does*

$$\binom{2^{\aleph_0}}{\aleph_0} \not\to \binom{2^{\aleph_0}}{\aleph_0} \wedge \binom{2^{\aleph_0}}{\aleph_0}$$

hold if we do not assume C.H.?

ADDED IN PROOF (May, 1970). We learned from a letter of F. Galvin that Laver proved that

$$\binom{\aleph_1}{\aleph_0} \to \binom{\aleph_1}{\aleph_0}, \binom{\aleph_1}{\aleph_0}$$

is consistent with ZF + AC.

C.H. implies even

$$\binom{\aleph_1}{\aleph_1} \not\to \binom{\aleph_1}{\aleph_0} \vee \binom{\aleph_0}{\aleph_1}, \binom{\aleph_1}{\aleph_0} \vee \binom{\aleph_0}{\aleph_1}.$$

We did not investigate the case $c \geq \aleph_0$ in detail. We mention that a surprising number of set theoretical problems can be formulated with the help of the polarized partition symbol.

We mention, e.g. that

$$\binom{\aleph_2}{\aleph_1} \not\to \binom{2}{\aleph_1}_{\aleph_0}$$

is equivalent to Problem 19/A.

The following slightly weaker statement seems to be of type (A) as well.

Problem 27. *Assume G.C.H. Does then*

$$\binom{\aleph_2}{\aleph_1} \nrightarrow \binom{\aleph_0}{\aleph_1}_{\aleph_0}$$

hold?

Note that the methods of [13] give the following results: G.C.H. $\Rightarrow$

$$\binom{\aleph_2}{\aleph_1} \nrightarrow \left(\begin{matrix}\aleph_2\\ \aleph_1\end{matrix}, \begin{matrix}\aleph_0\\ \aleph_1\end{matrix}\right)$$

$$\binom{\aleph_2}{\aleph_1} \rightarrow \left(\begin{matrix}\aleph_2\\ \aleph_1\end{matrix}, \binom{\aleph_2}{\aleph_0}_{\aleph_0}\right)$$

$$\binom{\aleph_2}{\aleph_2} \rightarrow \left(\begin{matrix}\aleph_2\\ \aleph_1\end{matrix} \vee \begin{matrix}\aleph_1\\ \aleph_2\end{matrix}, \begin{matrix}\aleph_1\\ \aleph_1\end{matrix}\right).$$

We never investigated Symbol-V as defined in (2). The methods of [13] would give a positive relation if the cardinals $a_0, \ldots, a_{l-1}$ are large and far from each other but the other cases must lead to complicated and involved problems.

We know, e.g. from a result of Sierpiński [35] that

$$\begin{pmatrix}\aleph_1\\ \aleph_1\\ \aleph_1\end{pmatrix} \nrightarrow \begin{pmatrix}\aleph_0\\ \aleph_0\\ \aleph_0\end{pmatrix}_2^{1,1,1}$$

holds. We state the following problem of type (D).

Problem 28.

$$\begin{pmatrix}\aleph_1\\ \aleph_1\\ \aleph_1\end{pmatrix} \rightarrow \begin{pmatrix}\aleph_0\\ \aleph_0\\ \aleph_0\end{pmatrix}_2^{1,1,1} ?$$

8. **Further problems on partitions.** Let S be a set and let $(S_\xi)_{\xi<\varphi}$ be a disjoint partition of it. Let $\Delta = (\mathscr{T}_\nu)_{\nu<\Omega(c)}$ be an r-partition of S. Δ is said to be canonical with respect to $(S_\xi)_{\xi<\varphi}$, if $X, Y \in [S]^r$, $|X \cap S_\xi| = |Y \cap S_\xi|$ for every $\xi < \varphi$ implies that $X \in \mathscr{T}_\nu$ and $Y \in \mathscr{T}_\nu$ are equivalent for every $\nu < \Omega(c)$.

Problem 29. *Assume G.C.H. Let* $|S| = \aleph_\omega$, $(S_n)_{n<\omega}$ *a disjoint partition of* S, $|S_n| = \aleph_n$. *Let further* $\Delta_r = (\mathscr{T}_0^r, \mathscr{T}_1^r)$ *be an r-partition of type 2 of S for every* $r < \omega$.

Does there then always exist an $S' \subset S$, $|S'| = \aleph_\omega$ *such that* Δ_r *is canonical with respect to* $(S_n')_{n<\omega}$ *for every* $r < \omega$ *(where* $S_n' = S' \cap S_n$*)?*

This should be compared with Lemma 3 of [13]. Note that one can ask the problem without assuming G.C.H. for a singular strong limit cardinal instead of

$\aleph_\omega$ and a corresponding version of Lemma 3 [13] remains valid for singular strong limit cardinals.

Let

$$a \to \left(\binom{b}{c}, d\right)^2$$

denote that the following statement is true:

Whenever $|S| = a$ and $(\mathscr{T}_0, \mathscr{T}_1)$ is a 2-partition of type 2 of S, then either there exist S', S'', $S' \cap S'' = 0$, $|S'| = b$, $|S''| = c$ such that $x \in S'$, $y \in S''$ implies $\{x, y\} \in \mathscr{T}_0$ or there exists $S' \subset S$, $|S'| = d$ such that $[S']^2 \subset \mathscr{T}_1$. It is obvious that a more general symbol corresponding to Smybol-I can be defined and with the help of the methods of [13] we can discuss almost all the problems, e.g.

$$\text{G.C.H.} \Rightarrow \aleph_\omega \to \left(\binom{\aleph_\omega}{\aleph_\omega}, \aleph_\omega\right)^2$$

holds. We mention one which remains unsolved.

Problem 30. *Assume G.C.H. Does*

$$\aleph_{\omega+1} \not\to \left(\binom{\aleph_{\omega+1}}{\aleph_\omega}, \aleph_1\right)^2$$

hold?

Note that as a corollary of Theorems I, 10 of [13] we have

$$\text{G.C.H.} \Rightarrow \left(\begin{matrix}\aleph_{\omega+1} \to (\aleph_{\omega+1}, \aleph_0)^2 \\ \aleph_{\omega+1} \not\to (\aleph_{\omega+1}, \aleph_1)^2\end{matrix} \quad \text{and} \quad \aleph_{\omega+1} \not\to \left(\binom{\aleph_{\omega+1}}{\aleph_\omega}, \aleph_2\right)^2\right).$$

Problem 31. *Can one prove without assuming G.C.H. that*

$$2^{\aleph_0} \not\to \left(\binom{\aleph_1}{\aleph_0}, \binom{\aleph_1}{\aleph_0}\right),$$

or at least

$$2^{\aleph_0} \not\to \left(\binom{\aleph_1}{\aleph_1}, \binom{\aleph_1}{\aleph_1}\right)$$

holds?

Problem 32. *Assume G.C.H. Does there exist a graph $\mathscr{G} = \langle g, G\rangle$ with $|g| = \aleph_1$, not containing a subgraph of type $[\aleph_0, \aleph_1]$ for which there exist a set $|S| = \aleph_1$ and a 2-partition $(\mathscr{T}_0, \mathscr{T}_1)$ of S such that neither the graph $(S, \mathscr{T}_0)$ nor the graph $(S, \mathscr{T}_1)$ contains a subgraph isomorphic to $\mathscr{G}$*?

This could be expressed by $\aleph_1 \not\to (\mathscr{G}, \mathscr{G})^2$ and would be a further strengthening of the relation

$$\aleph_1 \not\to \left(\binom{\aleph_1}{\aleph_0}, \binom{\aleph_1}{\aleph_0}\right).$$

(For the graph terminology used here see, e.g. [**10**].)

9. **Problems on set mappings.** Let S be a set. A function f with domain $[S]^a$ or $[S]^{<a}$ and with $f(X) \subset S - X, |f(X)| < b$ is said to be a set mapping on S of order $\leq b$ and of type a $(<a)$ respectively.

$S' \subset S$ is said to be a free subset if $f(X) \cap S' = 0$ for $X \in [S']^a$ (or $X \in [S']^{<a}$), respectively.

DEFINITION OF SYMBOL-VI. $(m, a, b) \xrightarrow{\text{VI}} n$ (or $(m, <a, b) \xrightarrow{\text{VI}} n$) is said to hold if for every S, $|S| = m$ and for every set mapping of type a (of type $<a$) and order $\leq b$ there exists a free subset $S' \subset S$, $|S'| = n$.

We introduced set mappings of type >1 in [**1**]. We will point out some problems stated in [**1**].

In [**1**] Theorem 7 we proved that $(m, <\aleph_0, b) \xrightarrow{\text{VI}} m$ holds for $b < m$ provided m is 0-1 measurable. In view of the recent results one can expect a positive answer to

Problem 33. *Does* $(m, <\aleph_0, 2) \overset{\text{VI}}{\not\to} \aleph_0$ *or at least* $(m, <\aleph_0, \aleph_0) \overset{\text{VI}}{\not\to} \aleph_0$ *hold for those* m *for which* $m \overset{\text{IV}}{\not\to} (\aleph_0)_2^{<\aleph_0}$ *holds?*

As a matter of fact we could not prove this even for $m = \aleph_\omega$.

See Problem 1 of [**1**].

This problem is also relevant to Jónsson's problem, see [**9**, p. 22].

We know that $m \xrightarrow{\text{I}} (n)_b^k$ implies $(m, k, b^+) \xrightarrow{\text{VI}} n$ but here we do know that the positive results thus obtained are the best possible.

Problem 34.

(A) *Assume that* m *is regular and* $m \overset{\text{I}}{\not\to} (m, m)^2$. *Does then* $(m, 2, 2) \overset{\text{VI}}{\not\to} m$ *hold?*

(B) *Assume G.C.H. Does then*

$$(\aleph_2, 3, 2) \not\to \aleph_1 \quad or \quad (\aleph_3, 3, \aleph_0) \not\to \aleph_2$$

hold?

(Note that $m \not\to (m, m)^2$ implies $m \not\to (m, 4)^3$ [**27**] and $\aleph_2 \not\to (\aleph_1)_2^3$, $\aleph_3 \not\to (\aleph_2)_{\aleph_1}^3$ holds if G.C.H. is assumed. See [**13**].) If m is a singular strong limit cardinal then $m \not\to (m, m)^2$ but $(m, 2, 2) \to m$.

Problem 35 (Hajnal). *Assume G.C.H. Let* S *be a set* $|S| = \aleph_{\omega+1}$. *Let* f *be a set mapping on* S *of type* 1 *and order* $\leq \aleph_{\omega+1}$. *Assume further that* $|f(x) \cap f(y)| < \aleph_\omega$ *for every pair* $x \neq y \in S$.

Does there exist a free subset of power $\aleph_1$?

In [**25**, Theorem 1] assuming G.C.H. this problem is settled in the negative for cardinals $\aleph_{\alpha+1}$ where $\aleph_\alpha$ is regular ($\aleph_1$ stands for $\aleph_{\text{cf}(\omega)+1}$).

Problem 36 (Hajnal). *Let $S, \prec$ be an ordered set of type ω_1. Let f be a set mapping on S of order $\leq \omega_1$ and of type 1. Assume further that $|f(x) \cap f(y)| < \aleph_0$ for every pair $x \neq y \in S$.*

Let $\alpha < \omega_1$. Does there then exist a free subset S', such that typ $S'(\prec) = \alpha$?

Note that the answer is positive if $\alpha < \omega^2 \cdot 2$. (See [25].)

The following problem on almost disjoint sets is strongly connected to the problems mentioned above

Let $|S| = b$; does there exist a system $\mathcal{F} \subset [S]^a$ such that $\mathcal{F}$ is almost disjoint, i.e. $A, B \in \mathcal{F}$, $A \neq B$ implies $|A \cap B| < a$ and such that for $S' \subset S$, $|S'| = a^+$ there is an $A \in \mathcal{F}$, $A \subset S'$? It was proved in [25] that the answer is yes if $a \geq \aleph_0$ is regular and $b = a^+$ and G.C.H. holds. The simplest unsolved problems are

Problem 37 (Hajnal). *What is the answer to the above stated problem if G.C.H. holds and*

(A) $a = \aleph_\omega$, $b = \aleph_{\omega+1}$,

(B) $a = \aleph_0$, $b = \aleph_2$?

Many special and difficult problems arise if we consider set mappings of type 1 on the set R of real numbers under different conditions imposed on the sets $f(x)$.

Here are some typical unsolved ones.

Problem 38. *Let f be a set mapping of type* 1 *on R.*

(A) *Assume that f is nowhere dense in R. Does there then exist a free subset of power $\aleph_1$?*

(B) *Let f be closed and of measure ≤ 1. Does there then exist a free subset of at least* 3 *points?*

(C) *Let f be bounded and of outer measure ≤ 1. Does there then exist an infinite independent set?*

Remarks. In case (A) A. Máté [31] proved that for every $\alpha < \omega_1$ there exists an independent set of type α. We do not even know the answer in case (A), if $f(x)$ is an ω sequence with limit point x for every x.

In case (B), Gladysz [24] proved that there is an independent pair.

In case (C) we proved in [2] that for every $k < \omega$ there exists an independent set of k elements, but an independent set of power $\aleph_1$ does not necessarily exist.

10. **Problems on families of sets stated in [4].**

Definition. A family $\mathcal{F}$ is said to have *property* B if there is a set B such that $A \cap B \neq 0$ and $A \cap -B \neq 0$ for every $A \in \mathcal{F}$. $\mathcal{F}$ is said to *have property* B(s) if there is a set B such that $1 \leq |A \cap B| < s$ for every $A \in \mathcal{F}$.

Problem 39. *Assume G.C.H. Let $|S| = \aleph_{\omega+1}$, $\mathcal{F} \subset [S]^{\aleph_1}$ and assume $|A \cap B| < \aleph_0$ for every pair $A \neq B \in \mathcal{F}$.*

Does then $\mathcal{F}$ possess property B($\aleph_1$) *or at least property* B?

The statement is true if $\aleph_{\omega+1}$ is replaced by a smaller cardinal.

For the background see [4]. We think that this is a problem of type (B) if not of type (A).

The following problem seems to be of type (A). There are many possible versions in which to formulate it. One of these is the following.

Problem 40. *Is it true that every $\mathcal{F}$, with $|\mathcal{F}| < 2^{\aleph_0}$, $\mathcal{F} \subset [S]^{\aleph_0}$ has property* B?

(If C.H. holds, the answer is obviously yes.)

In [4] we have stated the following problem (7):

Assume $\mathcal{G} = \langle g, G \rangle$ is a graph of $\aleph_2$ vertices. Suppose that every subgraph $\mathcal{G}'$ of it spanned by at most $\aleph_1$ vertices has chromatic number $\leq \aleph_0$. Does then $\mathcal{G}$ have chromatic number $\leq \aleph_0$? Recently in [12] we proved that the answer is negative.

We have proved assuming G.C.H. that there exist graphs $\mathcal{G}_{k+1}$ of power $\aleph_{k+1}$ for every $k < \omega$ such that every subgraph of $\mathcal{G}_{k+1}$ spanned by at most $\aleph_k$-vertices has chromatic number $\leq \aleph_0$, but $\mathcal{G}_{k+1}$ has chromatic number greater than $\aleph_0$. (In fact, we prove a more general result which can even be formulated without G.C.H.)

The following problems remain open:

Problem 41. *Assume G.C.H.*

(A) *Does there exist a graph $\mathcal{G}$ of $\aleph_{\omega+1}$ vertices, with chromatic number $> \aleph_0$, such that every subgraph $\mathcal{G}'$ spanned by less than $\aleph_{\omega+1}$ vertices has chromatic number at most $\aleph_0$?*

(B) *Does there exist a graph $\mathcal{G}$ with $\aleph_2$ vertices with chromatic number $\aleph_2$ such that each subgraph spanned by less than $\aleph_2$ vertices has chromatic number $\leq \aleph_0$?*

Note that a corresponding genuine problem can easily be formulated without G.C.H. It is quite possible that the answer to 41/A is yes if $\aleph_{\omega+1}$ is replaced by any regular cardinal a which is not too large ($a \in C_0 \wedge [\aleph_1 a] \subseteq C_1$ of Keisler-Tarski [30]).

Our methods of [12] break down for some very similar problems stated in [4].

Problem 42. *Assume G.C.H.*

(A) *Does there exist a family $\mathcal{F}$, $|\mathcal{F}| = \aleph_2$, $\mathcal{F} \subseteq [S]^{\aleph_0}$ for some S, such that if $\mathcal{F}' \subset \mathcal{F}$, $|\mathcal{F}'| = \aleph_1$ then $\mathcal{F}'$ possesses property* B *and $\mathcal{F}$ does not possess property* B?

(B) *Does there exist a graph $\mathcal{G}$ of power $\aleph_2$ such that every subgraph $\mathcal{G}'$ spanned by less than $\aleph_2$ vertices can be directed so that the number of directed edges emanating from a vertex is finite for every vertex, but this is no longer true for the graph $\mathcal{G}$?*

(C) (W. Gustin) *A family $\mathcal{F}$ is said to have property* G *if there exists a function f with $D(f) = \mathcal{F}$ such that $f(F) \in \mathcal{F}$ and $f(A) \neq f(B)$ for every pair $A \neq B \in \mathcal{F}$.*

Does there exist a family $\mathcal{F}$, $|\mathcal{F}| = \aleph_2$, $\mathcal{F} \subset [S]^{\aleph_0}$ such that every subfamily $\mathcal{F}' \subset \mathcal{F}$, $|\mathcal{F}'| < \aleph_2$ has property G, *but F does not possess property* G?

We mention that a number of related problems are stated in [4] which we do not repeat here. Note that in view of the results stated in [28] and [30] Problem 10 of [4] has already been solved. We state one more problem of [4] which seems to be of type (A) but of quite different character.

Problem 43. *Let S be the set of ordinals less than ω_1. Does there exist a function f with $D(f) = \omega_1$, $R(f) \subset \omega_1$ such that $f(\xi) < \xi$ for every $\xi < \omega_1$,* *and for every limit number ξ there exists an increasing sequence of ordinals ξ_n, $n < \omega$, $\lim \xi_n = \xi$ such that $\xi_n = f(\xi_{n+1})$ for $n < \omega$?*

ADDED IN PROOF. J. E. Baumgartner proved that the answer to Problem 43 is affirmative, and that it is certainly not of type (A).

11. Problems on chromatic and colouring numbers of graphs [10], [11]. For the results underlying Problems 44–50 see [10] and for the rest see [11].

Problem 44. *Assume C.H. Does there exist a graph $\mathcal{G}$ with $\aleph_1$ vertices of chromatic number $\aleph_1$ which does not contain a complete graph of 3 elements and does not contain a complete even graph $[\![\aleph_0, \aleph_0]\!]$?*

ADDED IN PROOF (May, 1970). A positive answer is given in Hajnal's paper mentioned on p. 24.

(A complete even graph $[\![a, b]\!]$ consists of two disjoint sets $|A| = a$, $|B| = b$ and of the edges with one endpoint in A and one endpoint in B.) The answer is affirmative by [10, 5.9] if 3 is replaced by $\aleph_0$.

A positive answer would be implied by

Problem 45.

(A) *Assume $\mathcal{G}$ is a graph of chromatic number $a \geq \aleph_0$. Does then $\mathcal{G}$ contain a subgraph $\mathcal{G}'$ of chromatic number a such that $\mathcal{G}'$ does not contain a triangle?*

(This might be a problem of type (D).)

(B) *Does there exist a function $f(k) < \omega$, for $k < \omega$ such that $f(k) \to +\infty$ if $k \to +\infty$ and such that every graph with chromatic number $\geq k$ contains a subgraph of chromatic number $\geq f(k)$ not containing triangles?*

Problem 46. *Let $\mathcal{G}$ be a graph of chromatic number greater than $\aleph_0$. Does then $\mathcal{G}$ contain odd circuits of length $2i + 1$ for $i > j$ for some $j < \omega$?*

The answer is affirmative if $\mathcal{G}$ has chromatic number greater than $\aleph_1$.

Problem 47. *Let $\mathcal{G}$ be a graph of chromatic number $\aleph_0$ and put $N = \{i < \omega$: there is a circuit of length i contained in $\mathcal{G}\}$. Is it true that*

$$\sum_{i \in N} \frac{1}{i} = +\infty?$$

DEFINITION. The *colouring number of a graph* $\mathcal{G}$ is the smallest cardinal b for which the set of vertices has a well-ordering $\prec$ satisfying

$$|\{y \prec x : y \text{ is connected to } x \text{ in } \mathcal{G}\}| < b$$

for every vertex x.

The colouring number of a graph is greater than or equal to its chromatic number.

The problem involved in the Symbol-VII to be defined is due to R. Rado.

DEFINITION OF SYMBOL-VII. $(a, b) \xrightarrow{\text{VII}} (c, d)$ is said to hold if every graph $\mathcal{G}$ with a vertices, all whose subgraphs spanned by a set of power $<b$ have colouring number $\leq c$, has colouring number $\leq d$.

In [10] we prove several results concerning this symbol.

Problem 48. *Assume G.C.H.*

(A) $(\aleph_2, \aleph_2) \overset{\text{VII}}{\not\longrightarrow} (\aleph_0, \aleph_0)$?

(B) $(\aleph_{\omega+1}, \aleph_2) \overset{\text{VII}}{\not\longrightarrow} (\aleph_0, \aleph_1)$?

Here $(\aleph_2, \aleph_2) \overset{\text{VII}}{\longrightarrow} (\aleph_0, \aleph_1)$ is true.

This should be compared with Problem 41/A.

We prove in [10, Theorems 9.1 and 10.1]

$$(a, \aleph_0) \overset{\text{VII}}{\longrightarrow} (k, 2k-2) \quad \text{for} \quad 2 \leq k < \omega, \quad a \text{ arbitrary}$$

but

$$(\aleph_0, \aleph_0) \overset{\text{VII}}{\not\longrightarrow} (k, 2k-3) \quad \text{and} \quad (\aleph_n, \aleph_n) \overset{\text{VII}}{\not\longrightarrow} (k, 2k-3), n < \omega$$

provided G.C.H. holds.

Problem 49. *Assume G.C.H. Is* $(\aleph_{\omega+1}, \aleph_{\omega+1}) \overset{\text{VII}}{\longrightarrow} (k, l)$ *true for some* $l < 2k-2$?

Problem 50. *Assume G.C.H. Is it true that if* $\mathscr{G}$ *has* $\aleph_{\omega+1}$ *vertices and does not contain a complete even graph* $[\![\aleph_0, \aleph_2]\!]$ *then* $\mathscr{G}$ *has colouring number* $\leq \aleph_1$?

This is true if $\aleph_{\omega+1}$ is replaced by a smaller cardinal a. This should be compared with Problem 39, since the method for proving the theorems for $a < \aleph_{\omega+1}$ is very similar.

We turn to graph decomposition problems considered in [11].

DEFINITION OF SYMBOL-VIII. Let $\mathscr{G} = \langle g, G \rangle$ be a graph with set of vertices g and with set of edges $G \subset [g]^2$.

The sequence $\mathscr{G}_\xi = \langle g_\xi, G_\xi \rangle$, $\xi \prec \varphi$, is said to be an edge decomposition of type $|\varphi|$ of $\mathscr{G}$ if $g_\xi = g$ and $\bigcup_{\xi<\varphi} G_\xi = G$. (An edge decomposition of a complete graph $G = \langle g, [g]^2 \rangle$ is a two partition of the set g.) $(a, b) \overset{\text{VIII}}{\longrightarrow} (c, d)$ denotes that every graph $\mathscr{G}$ of a vertices not containing a complete subgraph of power b has an edge decomposition $\mathscr{G}_\xi$, $\xi < \Omega(c)$, of type C where the members $\mathscr{G}_\xi$ do not contain a complete d graph.

Though the problems seem to be quite fundamental our results are very sketchy. We know, e.g.

$$((2^{(2^{\aleph_0})^+})^+, \aleph_0) \overset{\text{VIII}}{\not\longrightarrow} (\aleph_0, k)$$

holds for every $k < \omega$ [i.e. G.C.H. $\Rightarrow (\aleph_4, \aleph_0) \overset{\text{VIII}}{\not\longrightarrow} (\aleph_0, k)$, $k < \omega$], but probably the relation in the following problem is true.

Problem 51. $((2^{\aleph_0})^+, \aleph_0) \overset{\text{VIII}}{\not\longrightarrow} (\aleph_0, k)$ *for* $k < \omega$?

(Note that $(2^{\aleph_0}, (2^{\aleph_0})^+) \overset{\text{VIII}}{\longrightarrow} (\aleph_0, 3)$ is trivial since $2^{\aleph_0} \overset{\text{I}}{\not\longrightarrow} (3)^2_{\aleph_0}$ holds.)

Problem 52. $((2^{\aleph_0})^+, \aleph_1) \overset{\text{VIII}}{\longrightarrow} (\aleph_0, \aleph_0)$?

We do not know if $(m, \aleph_1) \overset{\text{VIII}}{\not\longrightarrow} (\aleph_0, \aleph_0)$ holds for any $m > 2^{\aleph_0}$. So it might be that every graph not containing a complete $\aleph_1$ graph can be decomposed into the union of $\aleph_0$ graphs not containing complete $\aleph_0$ graphs. However, this seems to be very unlikely. We suspect that $((2^{\aleph_0})^+, \aleph_1) \overset{\mathrm{VIII}}{\not\longrightarrow} (\aleph_0, \aleph_0)$ holds or at least assuming G.C.H. one can prove $(\aleph_2, \aleph_1) \overset{\mathrm{VIII}}{\not\longrightarrow} (\aleph_0, \aleph_0)$.

Note that $\aleph_1$ has a special role in Problem 52. We know, e.g. that

$$((2^{\aleph_0})^+, (2^{\aleph_0})^+) \overset{\mathrm{VIII}}{\not\longrightarrow} (\aleph_0, \aleph_1)$$

holds; i.e. assuming G.C.H.

$$(\aleph_2, \aleph_2) \overset{\mathrm{VIII}}{\not\longrightarrow} (\aleph_0, \aleph_1),$$

and trivially

$$(\aleph_2, \aleph_3) \overset{\mathrm{VIII}}{\longrightarrow} (\aleph_1, 3).$$

We do not know the answer for the following.

Problem 53. *Does* $(m, k+1) \overset{\mathrm{VIII}}{\not\longrightarrow} (\aleph_0, k)$ *hold for any* $m > 2^{\aleph_0}$, $3 \leq k < \omega$? (*Does* $((2^{\aleph_0})^+, 4) \overset{\mathrm{VIII}}{\longrightarrow} (\aleph_0, 3)$ *hold*?)

There is a very interesting finite problem here. It is obvious that $(a, b^+) \overset{\mathrm{VIII}}{\not\longrightarrow} (c, d)$, $a \geq b$ holds if $b \overset{\mathrm{I}}{\longrightarrow} (d)^2_c$ is true. One can ask if this is a best possible condition if a, b, c, d are finite. This is certainly not so, since 6 is the smallest number for which $b \overset{\mathrm{I}}{\longrightarrow} (3)^2_2$ holds but there is an $a(2, 3) = a$ for which $(a, 4) \overset{\mathrm{VIII}}{\not\longrightarrow} (2, 3)$ holds. (This was proved by Volkmann, but the involved proof is still unpublished, a is very large.)

It is reasonable to conjecture

Problem 54. *For every pair of integers* c, d *there exists an integer* $a(c, d)$ *such that*

$$(a(c, d), d+1) \overset{\mathrm{VIII}}{\not\longrightarrow} (c, d)$$

holds.

We cannot make any guess on the order of magnitude of $a(c, d)$.

12. **Problems of [14] and [8].** In [14] we consider several arrow relations of new type. We point out only one problem of the 15 problems stated there.

DEFINITION OF SYMBOL-IX. Let a, b, c, d be types or cardinals, e a cardinal but b, c, d are cardinals if a is a cardinal. Let S, $\prec$ be an ordered set of type a or a set of cardinal a respectively. $a \overset{\mathrm{IX}}{\longrightarrow} [b, c]^d_e$ is said to hold if for every family $\mathscr{F}$, $|\mathscr{F}| = e$ of subsets of S, either there is an S' of typ $S'(\prec) = b$ ($|S'| = b$) such that for every $X \subset S'$, typ $X(\prec) = d$ ($|X| = d$) there is an $A \in \mathscr{F}$, $X \subset A$ or there is an $S'' \subset S$, typ $S''(\prec) = c$ ($|S''| = c$) and an $\mathscr{F}' \subset \mathscr{F}$, $|\mathscr{F}'| = e$ such that $S'' \cap \bigcup \mathscr{F}' = 0$ respectively.

Problem 55. *Assume G.C.H.*

(A) $\aleph_2 \xrightarrow{\mathrm{IX}} [\aleph_2, a]^{\aleph_0}_{\aleph_1}$ $a = \aleph_1 \vee a = \aleph_2$?

(B) $\aleph_{\omega+1} \xrightarrow{\mathrm{IX}} [\aleph_{\omega+1}, \aleph_0]^{\aleph_\omega}_{\aleph_{\omega+1}}$?

(C) $\aleph_{\omega_1} \xrightarrow{\mathrm{IX}} [\aleph_{\omega_1}, \aleph_0]^{\aleph_0}_{\aleph_1}$?

Though these problems seem to be of type (E), because of the involved formulation, they are certainly difficult and, e.g. Problem 55/A might even be of type (A).

As usual the problems for types are more ramified and we do not have the space to discuss them here.

In [8] we considered problems of the following type.

Let S be a set $|S| = a$ and let f be a function defined on $[S]^k$, which associates a Lebesgue measurable subset $f(X)$ of $[0, 1]$ of Lebesgue measure $m(X) \geq u$ to every $X \in [S]^k$. For what type of subsets $\mathscr{T}$ of $[S]^k$ does there necessarily exist a $t \in [0, 1]$ such that $t \in \bigcap_{X \in \mathscr{T}} f(X)$?

We define a corresponding *Symbol*-X $(a, u)^k \xrightarrow{\mathrm{X}} \Delta$ where Δ stands for the corresponding class of subsets of $[S]^k$. We have genuine results only in case $k = 2$. We mention two of them:

$$(\aleph_0, u)^2 \xrightarrow{\mathrm{X}} [\![s+1]\!] \quad \text{iff} \quad u > 1 - \frac{1}{s} \quad \text{for} \quad 2 \leq s < \omega$$

where $[\![s+1]\!]$ stands for the class of complete subgraphs of $s + 1$ elements.

If $m > \aleph_0$ then $(m, u)^2 \xrightarrow{\mathrm{X}} \aleph_0$ for every positive u.

We can prove $(2^{\aleph_0}, u)^2 \overset{\mathrm{X}}{\not\longrightarrow} \aleph_1$ for $u \leq \frac{1}{2}$, but our proof for $(2^{\aleph_0}, u)^2 \overset{\mathrm{X}}{\not\longrightarrow} \aleph_1$ for u arbitrary uses C.H.

Problem 56. *Can one prove* $(2^{\aleph_0}, u)^2 \overset{\mathrm{X}}{\not\longrightarrow} \aleph_1$ *for some* $u > \frac{1}{2}$ *without using C.H.?*

It is clear from the remarks given in [8] that this problem is strongly connected with Problem 15.

Here are two other problems of [8] we are interested in.

Problem 57.

(A) $(\aleph_1, u)^3 \xrightarrow{\mathrm{X}} [\![4]\!]$ *for* $u > 0$,

(B) $(\aleph_0, u)^2 \xrightarrow{\mathrm{X}} [\![\aleph_0, \aleph_0]\!]$ *for* $u > \frac{1}{2}$,

where $[\![\aleph_0, \aleph_0]\!]$ *is a complete* $\aleph_0, \aleph_0$ *even graph?*

We know that (B) is false for $u \leq \frac{1}{2}$.

13. **Miscellaneous unpublished problems.**

Problem 58. *Let* S *be a set,* $|S| = a > 2^{\aleph_0}$. *Does there exist a disjoint* $\aleph_0$-*partition* $\bigcup_{\nu < \Omega(2^{\aleph_0})} \mathscr{T}_\nu = [S]^{\aleph_0}$ *of* S *satisfying the following condition?*

Whenever A_n, $n < \omega$ is a sequence of disjoint subsets of S, $|A_n| = 2$ for every $n < \omega$ then for every $\nu < \Omega(2^{\aleph_0})$ there is an $X \in \mathscr{T}_\nu$ such that X is a transversal of the sequence A_n, $n < \omega$.

If the answer is affirmative this is an improvement of the theorem mentioned before Problem 14/A. Note that for $a = 2^{\aleph_0}$ the answer is yes.

For $a > 2^{\aleph_0}$ we do not even know the answer for partitions of type 2.

ADDED IN PROOF. Kunen and F. Galvin proved that the answer to Problem 58 is affirmative for partitions of type 2 and of type $2^{\aleph_0}$ respectively.

DEFINITION OF SYMBOL-XI. Let S, $|S| = a$ be a set and let $\mathscr{A}$, $\mathscr{B}$ be classes of subgraphs of the complete graph with vertices S. $(a, b, \mathscr{A}) \xrightarrow{\mathrm{XI}} (c, \mathscr{B})$ is said to hold if the following statement is true.

Whenever $\mathscr{G}_\xi$, $\xi < \Omega(b)$ is a sequence of graphs $\mathscr{G}_\xi \in \mathscr{A}$ then there is a $\mathscr{G}^* \in \mathscr{B}$ and a set C of ordinals $<\Omega(b)\,|C| = c$ such that for $\xi \in C$, $\mathscr{G}^*$ and $\mathscr{G}$ have no common edges.

We have several unpublished results on Symbol-XI.

Let $\mathscr{A}(d, a)$ be the set of subgraphs of a complete graph of a vertices not containing complete d-graphs. Let $\mathscr{B}(d, a)$ be the set of complete subgraphs of a complete graph of a vertices spanned by d elements.

We can prove the following results:

$$(\aleph_1, \aleph_0, \mathscr{A}(3, \aleph_1)) \xrightarrow{\mathrm{XI}} (\aleph_0, \mathscr{B}(\aleph_0, \aleph_1)) \tag{1}$$

and

$$(\aleph_0, \aleph_1, \mathscr{A}(3, \aleph_1)) \xrightarrow{\mathrm{XI}} (\aleph_0, \mathscr{B}(\aleph_0, \aleph_0)) \tag{2}$$

but

$$(\aleph_1, \aleph_0, \mathscr{A}(3, \aleph_1)) \overset{\mathrm{XI}}{\not\longrightarrow} (\aleph_0, \mathscr{B}(\aleph_1, \aleph_1)) \tag{3}$$

provided C.H. holds.

The following seems to be an intriguing unsolved case.

Problem 59. *Does* $(\aleph_1, \aleph_0, \mathscr{A}(4, \aleph_1)) \xrightarrow{\mathrm{XI}} (\aleph_0, \mathscr{B}(\aleph_0, \aleph_1))$ *hold or does* $(\aleph_0, \aleph_1, \mathscr{A}(4, \aleph_0)) \xrightarrow{\mathrm{XI}} (\aleph_0, \mathscr{B}(\aleph_0, \aleph_0))$ *hold?*

We also do not know whether the relation in the following problem is true.

Problem 60. *Does* $(\aleph_0, \aleph_1, \mathscr{A}(\aleph_0, \aleph_0)) \overset{\mathrm{XI}}{\not\longrightarrow} (\aleph_0, \mathscr{B}(\aleph_0, \aleph_0))$ *hold?*

ADDED IN PROOF. Using the polarized partition relation V one can express Problems 59 and 60 as follows:

$$\begin{pmatrix}\aleph_0\\ \aleph_1\end{pmatrix} \to \begin{pmatrix}1 & \aleph_0\\ 4 & \aleph_0\end{pmatrix}^{1.2}$$

and

$$\begin{pmatrix}\aleph_1\\ \aleph_0\end{pmatrix} \to \begin{pmatrix}1 & \aleph_0\\ \aleph_0 & \aleph_0\end{pmatrix}^{1.2}$$

respectively.

The answer to both of these problems is affirmative. As to Problem 59 the following is true: If $a \geq \aleph_0$ is 0,1-measurable then

$$\begin{pmatrix}a\\ a^+\end{pmatrix} \to \begin{pmatrix}a & \vee & 1 & a\\ b & & a & a\end{pmatrix}^{1.2}$$

holds for every $b < a$.

The proof of this will be published in a forthcoming paper of A. Hajnal in the Fundamenta Mathematicae.

As to Problem 60 we proved that

$$\binom{a^+}{a} \to \binom{a}{a}_c^{1,2}$$

holds for every 0-1 measurable cardinal $a \geq \aleph_0$, $c < a$.

Then F. Galvin proved that

$$\binom{\aleph_1}{\aleph_0} \to \binom{\aleph_0}{\aleph_0}_c^{1,r}$$

holds for $r, c < \aleph_0$.

He conjectured that

$$\binom{a^+}{a} \to \binom{a}{a}_c^{1,<\aleph_0}$$

will hold for $c < a$ where a is 0-1 measurable cardinal greater than $\aleph_0$.

This was proved by A. Hajnal. For the proof see the above mentioned paper.

We give some more samples of the existing results.

Let $\mathscr{A}([\![a_1, a_2]\!], a)$ be the class of subgraphs of the complete graph $[\![a]\!]$ not containing complete even $[\![a_1, a_2]\!]$ graphs.

We have

$$(4) \qquad (\aleph_0, \aleph_0, \mathscr{A}([\![l, \aleph_0]\!])\aleph_0) \cap \mathscr{A}(k, \aleph_0)) \xrightarrow{\text{XI}} (\aleph_0, \mathscr{B}(\aleph_0, \aleph_0))$$

for every $l, k < \omega$ but a $\overset{\text{XI}}{\longleftrightarrow}$ relation holds with each of the classes standing on the right-hand side for every $l < \omega$ or $k < \omega$ respectively.

Let $\mathscr{B}([\![a_1, a_2]\!], a)$ be the class of complete even $[\![a_1, a_2]\!]$ subgraphs of the complete graph a.

We know

$$(5) \qquad (\aleph_1, \aleph_1, \mathscr{A}(\text{tree}, \aleph_1)) \overset{\text{XI}}{\longleftrightarrow} (\aleph_1, \mathscr{B}([\![\aleph_0, \aleph_0]\!], \aleph_1)).$$

But we do not know

Problem 61. $(\aleph_1, \aleph_0, \mathscr{A}(\text{tree}, \aleph_1)) \xrightarrow{\text{XI}} (\aleph_0, \mathscr{B}(\aleph_1, \aleph_1))$

$$or \xrightarrow{\text{XI}} (\aleph_0, \mathscr{B}([\![\aleph_0, \aleph_1]\!], \aleph_1))\,?$$

With $\mathscr{B}(\aleph_0, \aleph_1)$ we have an $\xrightarrow{\text{XI}}$ relation because of (1). (We do not even know the answer in case of disjoint trees.)

The results mentioned above will be published in a forthcoming paper by the two of us.

ADDED IN PROOF. Using C.H. the answer to Problem 61 is negative. In fact we have much stronger negative results. See our forthcoming paper mentioned above.

14. Miscellaneous problems continued. As a corollary of Lemma 5A of [13] we know that

$$2^{\cdot^{\cdot^{\cdot^{2^{\aleph_0}}}}}\Big\}{(r-1)} = \exp_{r-1}(\aleph_0) \nrightarrow (\aleph_1)_2^r \quad \text{for} \quad 2 \leq r < \omega.$$

We do not know

Problem 62. *Let $2 < r < \omega$, $|S| = \exp_{r-1}(\aleph_0)$. Does there exist an r-partition $(\mathscr{T}_0^r, \mathscr{T}_1^r)$ of type 2 of S satisfying the following conditions?*

(1) *$S' \subset S$, $|S'| = \aleph_1$ implies $[S']^2 \not\subset \mathscr{T}_i$ for $i < 2$ but*

(2) *$S' \subset S$, $|S'| = \aleph_1$ implies that for every $n < \omega$ and for every $i < 2$ there is an $S_i'' \subset S'$, $|S_i''| = n$ such that $[S_i'']^r \subset \mathscr{T}_i$.*

It is possible that (2) can be replaced by the stronger condition

(2′) *$S' \subset S$, $|S'| = \aleph_1$ implies that there are $S_i'' \subset S'$, $|S_i''| = \aleph_0$ such that $[S_i'']^2 \subset \mathscr{T}_i$ for $i < 2$.*

A positive answer to **Problem** 62 would be an improvement of the theorem of **[13]** already mentioned and it would be useful for the discussion of the following general problem which we formulated from an old result of W. Sierpinski **[36]**.

Definition of Symbol-XII. A family $\mathscr{F}$ of sets is said to have property $B(a, b)$, $b \geq 3$, if $\mathscr{F}' \subset \mathscr{F}$, $|\mathscr{F}'| = a$ implies that for every $b' < b$ there is an $\mathscr{F}'' \subset \mathscr{F}'$, $|\mathscr{F}''| = b'$, $\bigcap \mathscr{F}'' \neq 0$. ($B(a, 3)$ means that $\mathscr{F}$ does not contain a disjointed subfamily of power a.)

$(m, n) \xrightarrow{\text{XII}} (a, b)$ is said to hold if the Cartesian product of two families having property $B(a, b)$ has property $B(m, n)$.

It is easy to see that

$$(m, b^+) \xrightarrow{\text{XII}} (a, b^+) \tag{1}$$

is equivalent to

$$m \xrightarrow{\text{I}} (a)_2^b \quad \text{for} \quad b \leq \omega;$$

hence, e.g. $(2^{\aleph_0}, 3) \nrightarrow (\aleph_1, 3)$.

This was proved by Sierpiński and in fact his example gives $(2^{\aleph_0}, 3) \nrightarrow (\aleph_1, \aleph_1)$ but we do not know

Problem 63. *Assume $3 \leq r < \omega$.*

Is it true that

$$(\exp_{r-1}(\aleph_0), r+1) \nrightarrow (\aleph_1, \aleph_0)$$

or

$$(\exp_{r-1}(\aleph_0), r+1) \nrightarrow (\aleph_1, \aleph_1)$$

holds?

A positive answer to Problem 62 implies a positive answer to Problem 63.

Problem 64. *Do there exist two families $\mathscr{F}_1, \mathscr{F}_2$ both having property $B(\aleph_1, \aleph_0)$ and such that $\mathscr{F} = \mathscr{F}_1 \times \mathscr{F}_2$ does not possess property $B(\exp_k(\aleph_0), \aleph_0)$ for every $k < \omega$?*

We have some more partial results on the Symbol-XII which will be published later.

15. Miscellaneous problems continued.

Problem 65. *Let $|S| = \aleph_1$ and let $(\mathcal{T}_0, \mathcal{T}_1)$ be a 2 partition of type 2 of S. Let further S_n, $n < \omega$ be a disjointed sequence of subsets of S such that $|S_0| = \aleph_1$, $|S_n| = \aleph_0$ for $0 < n < \omega$. Does there then exist an increasing sequence of integers $(n_k)_{k<\omega}$, $n_0 = 0$ and a sequence B_k of subsets of S such that $|B_k| = \aleph_0$, $B_k \subset S_{n_k}$ and*

$$[B_0, B_k]^{1,1} = \{\{xy\} : x \in B_0 \wedge y \in B_k\} \subset \mathcal{T}_0$$

for some $i < 2$ and for every $1 < k < \omega$?

Problem 66 (Erdös, Hajnal, Milner). *Let $\mathcal{G} = \langle g, G\rangle$ be a graph and let $\prec$ be a well-ordering of the set g such that* typ $g(\prec) = \omega_1^\rho$, $\rho < \omega_2$. *Assume that $\mathcal{G}$ does not contain an infinite path. Does there then exist a subset $g' \subset g$,* typ $g'(\prec) = \omega_1^\rho$ *such that g' does not contain an edge?*

We know that the answer is affirmative for $\rho \leq \omega + 1$. The first unsolved case is $\rho = \omega + 2$.

In a forthcoming paper with E. Milner we will prove that if the condition that $\mathcal{G}$ does not contain an infinite path is replaced by the condition that $\mathcal{G}$ does not contain a quadrilateral then the answer is affirmative for every ρ even if ω_1 is replaced by ω_α.

The following simple problem seems to be strongly connected with well-known problems concerning denumerable order types.

Problem 67 (Erdös, Milner, Hajnal). *Let $\mathcal{G} = \langle g, G\rangle$ be a graph such that $|g| = \aleph_0$ and let $\prec$ be an arbitrary ordering of g.*

Assume $\mathcal{G}$ does not contain a quadrilateral. Put $\Theta =$ typ $g(\prec)$ *and assume that*

(a) $$\text{typ}(g - \{x\})(\prec) \geq \Theta \quad \textit{for every} \quad x \in g.$$

Does then g contain a subset g', typ $g'(\prec) = \Theta$ *such that g' contains no edge of $\mathcal{G}$?*

The problem whether for an arbitrary Θ there exist only finitely many vertices x which do not satisfy (a), is equivalent to the well-known problem whether a denumerable order type has only finitely many fixed points.

Added in proof (May, 1970). Laver proved that the answer to Problem 67 is affirmative.

Problem 68. *Assume G.C.H. Let S be a set $|S| = \aleph_1$ and let $[S]^2 = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$ be a 2-partition of type 3 of S. Assume $[S'] \subset \bigcup_{i \neq j < 3} \mathcal{T}_j$ implies $|S'| \leq \aleph_0$ for every $i < 3$. Does there then exist a subset $X \subset S$, $|X| = 3$ such that*

$$[X]^2 \cap \mathcal{T}_i \neq 0 \qquad \textit{for} \quad i < 3?$$

This problem is stated in [12] where several similar problems are formulated for the case $|S| = \aleph_1$. On the other hand, using the methods of [13] some general results can be proved which we preserve for later publication. We mention only one of them.

If $|S| = \aleph_1$ and $(\mathcal{T}_\nu)_{\nu<\omega_1}$ is a 2-partition of type ω_1 of S such that $[S']^2 \subset \bigcup_{\nu \neq \mu; \mu < \omega_1} \mathcal{T}_\mu'$ implies $|S'| \leq \aleph_0$ for $\nu < \omega_1$ then there is a subset $X \subset S$, $|X| = \aleph_0$ such that every pair of X belongs to different $\mathcal{T}_\nu$'s.

Problem 69. *Let $\mathscr{G} = \langle g, G \rangle$ be a graph $|g| = \aleph_1$. Assume that for every $g' \subset g$ there is a $g'' \subset g'$ such that g'' is finite and each vertex of $g' - g''$ is adjacent to at least one element of g''.*

Does then $\mathscr{G}$ contain a complete $\aleph_1$ graph?

Let $\mathscr{G} = \langle g, G \rangle$ be a graph and let $\mathscr{F}$ be a family of sets. We will briefly say that $\mathscr{F}$ *a-represents* $\mathscr{G}$ if there is a one-to-one mapping φ of g onto $\mathscr{F}$ such that

$$A \neq B \in g \quad \text{are connected in } \mathscr{G} \text{ iff } \quad |f(A) \cap f(B)| < a.$$

Assuming G.C.H. we can prove that if a is regular then every graph $\mathscr{G}$ of at most a^+ vertices can be a-represented by a family $\mathscr{F}$ of subsets of a set of power a.

We cannot answer

Problem 70. *Assume G.C.H. Let $\mathscr{G}$ be a graph of $\aleph_{\omega+1}$ vertices. Can it be $\aleph_\omega$-represented by a family $\mathscr{F}$ of subsets of a set of power $\aleph_\omega$?*

Let A be a set and $\mathscr{F}$ a family of subsets of A. Let $a = \{a_\xi\}_{\xi<\varphi}$ be a sequence of type φ of elements of A. $\mathscr{F}$ *strongly cuts a* if for every $\xi < \varphi$ there exists an $A_\xi \in \mathscr{F}$ such that $A_\xi \cap a = \{a_\zeta\}_{\zeta<\xi}$.

In [20] P. Erdös and M. Makkai proved that $|A| \geq \aleph_0$, $|\mathscr{F}| > A$ implies the existence of a sequence of type ω which is either strongly cut by $\mathscr{F}$ or is strongly cut by the family of the complements of $\mathscr{F}$ in A.

The following simple problems remain unsolved.

Problem 71 (Erdös, M. Makkai).

(A) *Assume $|A| = \aleph_1$, $|\mathscr{F}| > \aleph_1$. Does there then exist a sequence of type ω strongly cut by $\mathscr{F}$?*

(B) *Assume $|A| = \aleph_1$, $|\mathscr{F}| > \aleph_1$. Does there exist a sequence of length ξ, $\omega + 2 \leq \xi \leq \omega_1$ which is strongly cut either by $\mathscr{F}$ or by the family of the complements?*

ADDED IN PROOF (May, 1970). Recently S. Shelah obtained a number of results concerning this problem which we do not know yet in detail.

Problem 72. *Assume $\mathscr{G}$ with $|g| = \aleph_1$ does not contain a complete $\aleph_1$ graph. Does then its complement contain a topological complete $\aleph_1$ graph? See* [7].

DEFINITION OF SYMBOL-XIII. $a \xrightarrow{\text{XIII}} (b, c, d)$ is said to hold if the following statement is true. If $|S| = a$ and $\mathscr{F}$ is a family of subsets of S, such that $A \in \mathscr{F}$ implies $|A| < b$ and $A_1 \neq A_2 \in \mathscr{F}$ implies $A_1 \not\subset A_2$ then there are an $S' \subset S$ and $\mathscr{F}' \subset \mathscr{F}$ with $|S'| = c$, $|\mathscr{F}'| = d$ such that $S' \cap (\bigcup \mathscr{F}') = 0$.

Assuming G.C.H. we can give an almost complete discussion of this symbol and many results can be proved without assuming G.C.H. If $a \xrightarrow{\text{I}} (a, a)^2$ holds then $a \xrightarrow{\text{XIII}} (a, a, a)$ holds as well. The only genuine unsolved problem is the following.

Problem 73. *Assume a is strongly inaccessible and $a \overset{\mathrm{I}}{\nrightarrow} (a, a)^2$. Does then $a \overset{\mathrm{XIII}}{\longrightarrow\!\!\!\!\!\!\!+} \;(a, a, a)$ hold?*

Problem 74 (Erdös, Rado). *Assume G.C.H. Let $\mathscr{A}$ be the class of graphs of at most $\aleph_\omega$ vertices such that the valency of every vertex is less than $\aleph_\omega$. Does there exist a $\mathscr{G}_0 = \langle g_0, G_0\rangle \in \mathscr{A}$ such that every $\mathscr{G} \in \mathscr{A}$ is isomorphic to a subgraph of $\mathscr{G}_0$ spanned by some subset of g_0?*

Problem 75 (Erdös, Milner). *Assume G.C.H. Let $|S| = \aleph_\omega$ and let $\mathscr{F}$ be a family $\mathscr{F} \subset [S]^{\aleph_0}$, $|\mathscr{F}| = \aleph_{\omega+1}$. Does there then exist a disjoint partition $A \cup B \cup C = S$ of S such that $|C| \leq \aleph_0$ and both $A \cup C$ and $B \cup C$ contain $\aleph_{\omega+1}$ elements of $\mathscr{F}$?*

Problem 76 (P. Erdös). *Let $\mathscr{F}$ be a family of analytic functions in the unit circle so that for every Z, $|\{f(Z): f \subset \mathscr{F}\}| \leq a$. Is it true that $\mathscr{F}$ has power $\leq a$?*

This problem was asked for $a = \aleph_0$ by J. Wetzel, and P. Erdös proved that in this case the problem is equivalent to $2^{\aleph_0} > \aleph_1$. If $2^{\aleph_0} > a^+$, then the answer is affirmative in general. The real problem is, e.g. whether $2^{\aleph_0} = \aleph_2$ implies that the answer is negative with $a = \aleph_1$ (see [19]).

17. **Some problems in topology; a problem on generalized Ulam matrices.** The second author and I. Juhász considered several problems in general topology where the methods of combinatorial set theory could be applied. We state some of the unsolved problems which seem to be of purely set theoretical character, too.

Problem 77 (J. de Groot, B. A. Efimov, J. Isbell). *Does there exist a Hausdorff space of $(2^{\aleph_0})^+$ points not containing a discrete subspace of at least $\aleph_1$ points?*

The sharpest result is given in [29], a Hausdorff space of $(2^{2^{\aleph_0}})^+$ points contains a discrete subspace of $\aleph_1$ points. For references, see also [29].

Problem 77/A (A. Hajnal, I. Juhász). *Assume G.C.H. Let R be an ordered set $|R| = \aleph_2$ such that the character of every point of R is $\aleph_0$. Does there then exist a disjointed system $\mathscr{F}$ of power $\aleph_2$ of open intervals of R?*

Note that there is an obvious connection with a special case of the generalized Souslin problem.

Problem 78 (A. Hajnal, I. Juhász). *Does there exist a hereditarily separable Hausdorff space of cardinality greater than that of the continuum?*

Problem 79 (A. Hajnal, I. Juhász). *Assume G.C.H. Does there exist a regular space of power $\aleph_1$, such that each subspace of power $\aleph_1$ of it has weight $\aleph_2$?*

We proved this for Hausdorff spaces assuming G.C.H. See [29].

Problem 80 (A. Hajnal). *Let a be the first weakly inaccessible cardinal $>\aleph_0$.*

Let $|S| = a$. *Does there exist a triangular matrix* $A_{\xi,\eta}$ *of subsets of* S *for* $\xi < \eta < \Omega(a)$ *satisfying the following conditions:*

(1) *for every* $\xi < \Omega(a)$ *the family* $\{A_{\xi,\eta}\}_{\xi<\eta<\Omega(a)}$ *is disjointed?*

(2) *for every* $\eta < \Omega(a)$, $|S - \bigcup_{\xi<\eta} A_{\xi,\eta}| < a$?

This would be a straightforward generalization of Ulam matrices for inaccessible cardinals, and it would give a short direct proof of the fact, that there is an a complete field of sets generated by at most a elements containing $[S]^{<a}$ in which there is no a complete proper a-saturated ideal containing $[S]^{<a}$.

This statement holds for a wide class of weakly and strongly inaccessible cardinals.

In 1950, answering a problem of S. Ulam, L. Alaoglu and P. Erdös proved **[18]** that if $|S|$ is less than the first weakly inaccessible cardinal, then one cannot define $\aleph_0$ σ-additive 0-1 measures on S so that every subset of S is measurable with respect to one of them. It is obvious from their proof that as a corollary of recent results of R. Solovay this would hold if $|S|$ is even larger. Though we did not investigate the problem very closely, it might be worth mentioning that the following simple instance of the problem seems to be still unsolved.

Problem 81 (S. Ulam). *Let* $|S| = \aleph_1$. *Can one define* $\aleph_1$ σ*-additive* 0-1 *measures on* S *so that each subset is measurable with respect to one of them?*

We do not know what happens if 0-1 measure is replaced, e.g., by real valued measure.

Problem 82 (L. Gillman). *Let* $|S| = \aleph_1$ *and let* $\mathscr{T}$ *be a nonprincipal prime ideal in the set of subsets of* S. *Does there exist an* $\mathscr{T}' \subset \mathscr{T}$, $|\mathscr{T}'| = \aleph_1$ *such that* $\bigcup \mathscr{T}'' = S$ *for every* $\mathscr{T}'' \subset \mathscr{T}'$, $|\mathscr{T}''| \geq \aleph_0$?

References

1. P. Erdös and A. Hajnal, *On the structure of set mappings*, Acta Math. Acad. Sci. Hungar. 9(1958), 111–131.

2. ———, *Some remarks on set theory*. VIII, Michigan Math. J. 7 (1960), 187–191.

3. ———, *Some remarks on set theory*. VII, Acta Sci. Math. 21 (1960), 154–163.

4. ———, *On a property of families of sets*, Acta. Math. Acad. Sci. Hungar. 12 (1961), 87–123.

5. ———, *Some remarks concerning our paper "On the structure of set mappings"—Nonexistence of a two-valued σ-measure for the first uncountable inaccessible cardinal*, Acta Math. Acad. Sci. Hungar. 13 (1962), 223–226.

6. ———, *On a classification of denumerable order types and an application to the partition calculus*, Fund. Math. 5 (1962), 117–129.

7. ———, *On complete topological subgraphs of certain graphs*, Ann. Univ. Sci. Budapest 7 (1964), 143–149.

8. ———, *Some remarks on set theory*. IX, Michigan Math. J. 11 (1964), 107–127.

9. ———, *On a problem of B. Jónnson*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 14 (1966), 61–99.

10. ———, *On the chromatic number of graphs and set systems*, Acta Math. Acad. Sci. Hungar. 17 (1966), 61–99.

11. ———, *On decomposition of graphs*, Acta. Math. Acad. Sci. Hungar., **18** (1967)

12. ———, *On chromatic numbers of infinite graphs*,, Graph theory symposium held in Tihany, Hungary 1966 83–98.

13. P. Erdös, A. Hajnal, and R. Rado, *Partition relations for cardinals*, Acta Math. Acad. Sci. Hungar. **16** (1965), 193–196.

14. P. Erdös, A. Hajnal, and E. C. Milner, *On the complete subgraph of graphs defined by systems of sets*, Acta Math. Acad. Sci. Hungar., **17** (1966), 159–229.

15. J. Czipszer, P. Erdös, and A. Hajnal, *Some extremal problems on infinite graphs*, Publ. Math. Inst. Hungar. Acad. Sci. Ser. A **7** (1962), 441–457.

16. P. Erdös, G. Födor, and A. Hajnal, *On the structure of inner set mappings*, Acta Sci. Math **20** (1959), 81–90.

17. P. Erdös, A. Hajnal, and E. C. Milner, *On sets of almost disjoint subsets of a set*, Acta Math. Acad. Sci. Hungar, **19** (1968), 209–218.

18. P. Erdös, *Some remarks on set theory*, Proc. Amer. Math. Soc. **1** (1950), 133–137.

19. ———, *An interpolation problem associated with the continuum hypothesis*, Michigan Math. J. **11** (1964), 9–10.

20. P. Erdös and M. Makkai, *Some remarks on set theory*. X, Studia Sci. Math. Hungar. **1** (1966), 157–159.

21. P. Erdös and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. **62** (1956), 427–489.

22. ———, Partition relations and transitivity domains of binary relations formed LMS **42** (1967) 624–633.

23. P. Erdös and A. Tarski, *On families of mutually exclusive sets*, Ann. of Math. **44** (1943), 315–429. "On some problems involving inaccessible cardinals" in *Essays on the foundations of mathematics*, Magnes Press, Hebrew Univ., Jerusalem, 1961, pp. 50–82.

24. S. Gladysz, *Bemerkungen über die Unabhängigkeit der Punkte in Bezug auf Mengenwertigen Funktionen*, Acta Math. Acad. Sci. Hungar. **13** (1962), 199–201.

25. A. Hajnal, *Some results and problems on set theory*, Acta Math. Acad. Sci. Hungar. **11** (1960), 277–298.

26. ———, *Proof of a conjecture of S. Ruziewicz*, Fund. Math. **50** (1961), 123–128.

27. ———, *Remarks on a theorem of W. P. Hanf*, Fund. Math. **54** (1964), 109–113.

28. ———, *On the topological product of discrete spaces*, Notices Amer. Math. Soc. **11** (1964).

29. A. Hajnal and I. Juhász, *Discrete subspaces of topological spaces*, Indag. Math. **29** (1967), 343–356; *On hereditary α-Lindelöf and hereditary α-separable spaces*, Acta Math. Acad. Sci. Hungar., to appear.

30. H. J. Keisler and A. Tarski, *From accessible to inaccessible cardinals*, Fund. Math. **53** (1964), 225–308.

31. A. Máté, *On the theory of relations*, Publ. Inst. Hungar. Acad. Sci. **9** (1964), 331–333.

32. F. C. Milner and R. Rado, *The pigeonhole principle for ordinal numbers*, J. London Math. Soc. **15** (1965), 750–768.

33. F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. (2) **30** (1930), 264–286.

34. J. H. Silver, *Some applications of model theory in set theory*, Ph.D. Thesis, Univ. of California, Berkeley, 1966.

35. W. Sierpiński, *Sur quelques propositions concernant la puissance du continu*, Fund. Math. **36** (1951), 1–13.

36. ———, *Sur un problème de la théorie des relations*, Ann. Scuola Norm. Sup. Pisa **2** (1933), 285–287.

37. E. Specker, *Teilmengen von Mengen mit Relationen*, Comment. Math. Helv. **31** (1957), 302–314.

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