ON A PROBLEM OF GRÜNBAUM

BY

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In memory of my friend and collaborator, Leo Moser

\( P_n \) will denote a set of \( n \) points in the plane. A well known theorem of Gallai-Sylvester (see e.g. [4]) states that if the points of \( P_n \) do not all lie on a line then they always determine an ordinary line, i.e. a line which goes through precisely two of the points of \( P_n \).

Using this theorem I proved that if the points do not all lie on a line, they determine at least \( n \) lines. I conjectured that if \( n > n_0 \) and no \( n-1 \) points of \( P_n \) are on a line, they determine at least \( 2n-4 \) lines. This conjecture was proved by Kelly and Moser [3], who, in fact, proved the following more general result:

Let \( P_n \) be such that at most \( n-k \) of its points are collinear. Assume

\[ n \geq \frac{1}{2}(3(3k-2)^2+3k-1). \]

Then \( P_n \) determines at least

\[ kn - \frac{1}{2}(3k+2)(k-1) \]

lines. They also observed that (2) is best possible.

B. Grünbaum asked the following question: Determine the sequence of integers \( m^{(1)} < m^{(2)} < \cdots \) so that for every \( i \) there is a \( P_n \) which determines exactly \( m^{(i)} \) lines. \( m^{(2)} = 1, m^{(3)} = n, m^{(3)} = 2n-4 \) if \( n \geq 27 \) (see [3]). Clearly the largest value of \( m^{(n)} \) is \( \binom{n}{2} \). Grünbaum observed that \( \binom{n}{2} - 1 \) and \( \binom{n}{2} - 3 \) cannot be values of \( m^{(n)} \). The proof is easy. If the points are not in general position at least three must be on a line, thus \( m^{(n)} = \binom{n}{2} - 1 \) is impossible. If 4 points are on a line or there are two lines containing three points we get at most \( \binom{n}{2} - 5 \) or \( \binom{n}{2} - 4 \) lines, thus \( m^{(n)} = \binom{n}{2} - 3 \) is also impossible.

The problem of characterizing the sequence \( \{m^{(n)}\} \) seems to be very difficult. We prove the following

**THEOREM.** There exists \( c_1 \) such that for each \( m \) satisfying

\[ c_1 n^{3/2} < m \leq \binom{n}{2} \]

\( m \neq \binom{n}{2} - 1, m \neq \binom{n}{2} - 3 \), there is a \( P_n \) which determines exactly \( m \) lines.

We also show that our theorem is best possible in the following sense: There is a \( c_2 \) (\( c_1 \) and \( c_2 \) are absolute positive constants) so that there is an \( m > c_2 n^{3/2} \) for which there is no \( P_n \) which determines exactly \( m \) lines. To determine the largest

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such \( m \), seems to be a difficult problem; I doubt that the methods of this paper can solve it. In view of this we do not attempt to get the best values for \( c_1 \) and \( c_2 \).

First we show that there is an \( m > c_0 n^{3/2} \) so that no \( P_n \) determines \( m \) lines. Let \( k_0 \) be the largest integer for which

\[
 n > \frac{1}{4}(3k_0 - 2)^2 + 3k_0 - 1, \quad \text{i.e.} \quad k_0 = (1 + o(1)) \left( \frac{2n}{27} \right)^{1/2}.
\]

Put

\[
 m = k_0 n - \frac{1}{4}(3k_0 + 2)(k_0 - 1) - 1.
\]

It is easy to see that no \( P_n \) determines exactly \( m \) lines. If at most \( n-k_0 \) of the points lie on a line then by (2) \( P_n \) determines at least \( m+1 \) lines. Assume next that \( n-l, l < k_0 \) points of \( P_n \) are on a line. Then clearly \( P_n \) determines at most

\[
 1 + \left( \frac{l}{2} \right) + l(n-l), \quad l < k_0
\]

lines which by (3) and (4) is clearly less than \( m \) if \( n > n_0 \).

Now we prove our theorem. First we note the following

**Lemma.** Let \( c_1 \) be sufficiently large. Then every integer

\[
 t < \binom{n}{2} - c_1 n^{3/2}, \quad t \neq 1, \quad t \neq 3
\]

can be written in the form

\[
 t = \sum \alpha_i \left( \binom{n_i}{2} - 1 \right), \quad \sum \alpha_i n_i \leq n, \quad n_i \geq 3
\]

where the \( \alpha_i \) are positive integers.

Assume that our lemma has already been proved then we deduce our Theorem as follows:

Put \( m = \binom{n}{2} - t \). Our \( P_n \) which determines exactly \( m \) lines is constructed in the following way: \( P_n \) has \( \alpha_i \) lines \( i = 1, \ldots \) each of which has \( n_i \) points, otherwise the points are in general position, i.e. no three of them are on a line. It is clear by (6) that such a configuration exists and by (6) it determines

\[
 \binom{n}{2} - \sum \alpha_i \left( \binom{n_i}{2} - 1 \right) = m
\]

lines. Thus we only have to prove our lemma.

Let \( n_1 \) be the largest integer for which \( \binom{n_1}{2} < t - 4 \). Clearly \( n_1 \leq \sqrt{2t+1} < n - 10 \sqrt{n} \) for sufficiently large \( c_1 \), also

\[
 t - \binom{n_1}{2} < 3n_1 < 3n.
\]
Let now $n_2$ be the largest integer for which
\[
\binom{n_2}{2} \leq t - \binom{n_1}{2} - 4.
\]

Clearly $n_2 < 3\sqrt{n}$ and
\[
4 \leq t - \binom{n_1}{2} - \binom{n_2}{2} < 6\sqrt{n}.
\]

By (7) we can write
\[
t = \binom{n_1}{2} + \binom{n_2}{2} + \alpha_3 \left( \frac{4}{2} - 1 \right) + \alpha_4 \left( \frac{3}{2} - 1 \right)
\]
where $\alpha_3 + \alpha_4 < 3\sqrt{n}$. Thus (5) and (6) are satisfied and the proof of our lemma is complete.

It might be possible to determine the smallest $t$ which cannot be written in the form (6), but we do not discuss this question here.

I would like to say a few words about possible generalizations of our theorem. The following result is well known [2]:

Let $S$ be a set of $n$ elements $x_1, \ldots, x_n$. Suppose $A_i \subseteq S$, $2 \leq |A_i| < n$ ($1 \leq i \leq k$) and each pair $(x_r, x_s)$ ($1 \leq r, s \leq n$) is contained in exactly one $A_i$. Then $k \geq n$. Here I can prove that if
\[
n + cn_{1/4} < m \leq \binom{n}{2}, \quad m \neq \binom{n}{2} - 1, \quad m \neq \binom{n}{2} - 3
\]
then there are $m$ sets $A_i \subseteq S$, $2 \leq |A_i|$, so that every pair $(x_r, x_s)$ is contained in one and only one $A_i$. Probably $cn^{1/4}$ is best possible.

A straightforward application of our method leads to the following

**Theorem.** Let $cn^2 < m \leq \binom{n}{3}$, $m \neq \binom{n}{3} - a$, where $a$ runs through a finite set of numbers which could easily be determined explicitly. Then there is a $P_n$ which determines exactly $m$ circles. A recent result of Elliott [1] shows that the order of magnitude $cn^2$ is best possible.

**References**


2. As far as known to the author this result was first proved by Hanani in 1938 (he published his proof only later) and it was first published in N. G. de Bruijn and P. Erdös, *On a combinatorial problem*, Indig. Math. 10 (1948), 421–423.


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