On the Number of Unique Subgraphs of a Graph

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A subgraph $H$ of a graph $G$ is unique if $H$ is not isomorphic to any other subgraph of $G$. The existence of a graph on $n$ vertices having at least $2^{n^{3/4}-en^{9/8}}$ unique subgraphs is proven for $c > \frac{3}{2} \sqrt{2}$ and $n$ sufficiently large.

We will say a subgraph $H$ of a graph $G$ is unique if $H$ is not isomorphic to any other subgraph of $G$. In Figure 1 we give an example of a graph and its unique subgraphs.

![Figure 1](image-url)

Any graph $G$ with edges contains at least two unique subgraphs: $G$ itself and the graph obtained by deleting all edges of $G$. The complete graphs on more than one vertex have just two unique subgraphs.
We will show that, although a graph on $n$ vertices can have at most $2^{n \choose 2}$ subgraphs, still, for all large $n$, there are graphs with more than $2^{n^2/2-cn^{3/2}}$ unique spanning subgraphs where $c$ is any constant greater than $\frac{1}{3} \sqrt{2}$.

In the following an asymmetric graph is a graph with no non-trivial automorphism and $[x]$ and $\{x\}$ have their usual meanings as largest integer not more than $x$ and least integer not less than $x$, respectively.

We denote by $f(n)$ the largest number of unique-subgraphs a graph on $n$ vertices can have.

**Theorem.** $f(n) > 2^{n^2/2-cn^{3/2}}$ for $c > \frac{3}{\sqrt{2}}$ and $n$ sufficiently large.

**Proof.** We will construct a graph $G$ having the required number of unique subgraphs by constructing graphs $A$ and $B$ and then joining their vertices in a certain manner to form $G$.

We first construct a graph $A$ on $m = \left\lfloor \frac{-1 + \sqrt{8n + 1}}{2} \right\rfloor$ vertices so that the complement of $A$ is a tree $T$ having exactly one vertex $v$ of valence three and such that the removal of $v$ leaves three paths no two having the same length. Such a tree exists for $m \geq 7$ and hence for all $n \geq 28$. Clearly $T$, and so $A$, is asymmetric. For a later calculation we note that each vertex of $A$ has valence at least $m - 4$.

Next we construct a second graph $B$ with $n - m$ vertices by dividing these vertices as equally as possible into

$$k = \left\lfloor \frac{m}{2} \right\rfloor - 3$$

sets and joining by an edge any two vertices not in the same set. Again for future calculation we note, since there are at least

$$\left\lfloor \frac{n - m}{k} \right\rfloor$$

vertices in each set, that each vertex of $B$ has valence at most

$$n - m - \left\lfloor \frac{n - m}{k} \right\rfloor.$$

With each vertex $b$ of $B$ we associate a set $A_b$ of at least $m - 2$ vertices of $A$. Since $\binom{m}{2} + m + 1 \geq n - m$, the $A_b$ sets can be chosen to be distinct. Since also $\binom{m}{2} \leq n - m$ the $A_b$ sets can be chosen to include all
subsets of \( m - 2 \) vertices of \( A \) and hence so that any vertex of \( A \) is a member of at most one more \( A_b \) set than any other vertex of \( A \).

Now we let \( G \) be the graph consisting of \( A \) and \( B \) together with all edges joining each vertex \( b \) of \( B \) to all the vertices of \( A_b \). If \( a \) and \( b \) are vertices of \( A \) and \( B \), respectively, then, since there are at least \((n - m)(m - 2)\) such edges, \( a \) has valence at least

\[
\left\lfloor \frac{(n - m)(m - 2)}{m} \right\rfloor + m - 4
\]

and \( b \) has valence at most

\[
n - \left\lfloor \frac{n - m}{k} \right\rfloor
\]

so that \( a \) has greater valence than \( b \) if

\[
\left\lfloor \frac{n - m}{k} \right\rfloor > 2 + \frac{2n}{m}.
\]

It is easy to verify that this inequality holds for \( n \geq 1 \).

If the number of edges in \( B \) is \( t \) then, since

\[
\frac{n - m}{k} \leq \frac{2(n - m)}{m - 7} \leq \sqrt{2n} + 7,
\]

we have, for \( c > \frac{3}{2} \sqrt{2} \) and sufficiently large \( n \),

\[
t \geq \frac{1}{2} (n - m)(n - m - \left\lfloor \frac{n - m}{k} \right\rfloor) > \frac{1}{2} (n - \sqrt{2n})(n - 2\sqrt{2n} - 8) \\
\geq \frac{n^2}{2} - cn^{3/2},
\]

so that the proof will be complete if we show that any subgraph of \( G \) obtained by deleting edges of \( B \) is unique.

Suppose that \( H \) is any such subgraph and that \( H' \) is another subgraph of \( G \) isomorphic to \( H \) under \( \varphi \). \( \varphi \) must carry a vertex \( a \) of \( A \) to a vertex of \( A \) since the degree of \( a \) in \( H \) is larger than the degree in \( H' \) of any vertex of \( B \). The restriction of \( \varphi \) to \( A \) is then an automorphism on \( A \) and so each vertex of \( A \) is fixed by \( \varphi \) since \( A \) is asymmetric. Since this is so, for each vertex \( b \) of \( B \) we must have \( A_b = A_{\varphi(b)} \), which in turn requires \( \varphi(b) = b \), i.e., \( \varphi \) is the identity and \( H \) is unique.
We note that it follows from the proof that for proper choice of the constant \( c \) we have

\[
f(n) > 2^{n^{3/2} - 3\sqrt{2n^{3/2}} - e^n} \quad \text{for} \quad n \geq 1.
\]

It would be interesting to get a non-trivial upper bound for \( f(n) \). Perhaps our lower bound is close to being best possible but we have not even proved \( f(n) < 2^{n^{3/2} - n^{1+\varepsilon}} \) for a certain \( \varepsilon > 0 \).

The following question might also be of interest: Determine or estimate the largest \( r = r(n) \) so that there is a graph on \( n \) vertices in which the removal of \( r \) or fewer edges leaves a unique subgraph.