A New Function Associated with the Prime Factors of \((\varphi_k)\)

By E. F. Ecklund, Jr., P. Erdös and J. L. Selfridge

Abstract. Let \(g(k)\) denote the least integer > \(k + 1\) so that all the prime factors of \((\varphi_k)\) are greater than \(k\). The irregular behavior of \(g(k)\) is studied, obtaining the following bounds:

\[
g(k) < \exp(k(1 + o(1))).
\]

Numerical values obtained for \(g(k)\) with \(k \leq 52\) are listed.

The prime factors of \((\varphi)\) have been studied a great deal. In a recent paper, Erdös [2] stated several results and unsolved problems on this subject. In this paper, we discuss one of the problems stated there: Denote by \(g(k)\) the least integer > \(k + 1\) so that all prime factors of \((\varphi_k)\) are greater than \(k\). Determine or estimate \(g(k)\).

The behavior of \(g(k)\) is surprisingly irregular. We searched for values of \(g(k) \leq 2500000\) for \(2 \leq k \leq 100\); the results of this search are reported in Table 1. In reviewing Table 1, we noticed the surprising example \(g(28) = 284\). This motivated a second search for other such examples with \(g(k) \leq 100000\) and \(101 \leq k \leq 500\); none were found.

### Table 1. Values of \(g(k) \leq 2500000\) for \(2 \leq k \leq 100\)

<table>
<thead>
<tr>
<th>(k)</th>
<th>(g(k))</th>
<th>(k)</th>
<th>(g(k))</th>
<th>(k)</th>
<th>(g(k))</th>
<th>(k)</th>
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### \(B\): \(g(k)\) exceeds the search bound of 2500000

The following conjectures on \(g(k)\) all seem certainly true, and perhaps some of them will not be difficult to prove. First, we conjecture

\[
(1) \quad \limsup_{k \to \infty} \frac{g(k + 1)}{g(k)} = \infty \quad \text{and}
\]

\[
(2) \quad \liminf_{k \to \infty} \frac{g(k + 1)}{g(k)} = 0.
\]

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* The condition \(g(k) > k + 1\) was inserted to avoid the special case \(k + 1 = p\), a prime.
Also, it seems that \( g(k) \) is not of polynomial growth—in other words, for every \( n \) and \( k > k_0(n) \),

\[(3) \quad g(k) > k^r.\]

On the other hand,

\[(4) \quad \lim_{k \to \infty} g(k)^{1/k} = 1\]

certainly seems to hold, and we expect that

\[(5) \quad g(k) < \exp(c \pi(k))\]

is true.

We now give lower and upper bounds for \( g(k) \). For a lower bound, we show there is an absolute constant \( c > 0 \) such that

\[(6) \quad g(k) > k^2.\]

We first show that \( g(k) > 2k \) (for \( k > 4 \)) always holds. By definition, \( g(k) > k + 1 \), and \( g(k) \neq 2k \) since \( \binom{k}{t} \) is always even. Suppose \( g(k) = k + t \) with \( 1 < t < k \). We have \( \binom{k^+}{t} = \binom{k^+}{t} \). Ecklund [1] showed that \( \binom{k^+}{t} \) has a prime factor not exceeding \( (k + t)/2 < k \), the only exception being \( \binom{4}{3} \) which corresponds to the case \( k = 4 \), \( t = 3 \). Erdős and Selfridge [2, p. 406] proved that if \( m \geq 2k \), then \( \binom{n}{t} \) always has a prime factor \( < m/k^r \), for some absolute constant \( c > 0 \). This immediately implies (6).

Next, we give a very crude upper bound on \( g(k) \). Denote by \( L_k \) the least common multiple of the integers \( 1, 2, \ldots, k \) and put \( P = \prod_{p \leq k} p \). Let \( N(k, l) = L_k P \). If \( n + 1 \) is any multiple of \( N(k, l) \), then

\[(7) \quad g(k) < N(k, k) = \prod_{p \leq k} p^{\alpha_p + 1},\]

where \( \alpha_p = \lfloor \log_k n \rfloor \). For \( k > k_0 \), this upper bound can be improved a bit. We show

\[(8) \quad g(k) < k^2 L_k P \quad \text{with} \quad l = \lfloor 6k/\log k \rfloor.\]

To prove (8), consider the integers \( tL_k P - 1 \) for \( 1 \leq t \leq k^2 \). We show that, for at least one of these values of \( t \),

\[(9) \quad p \mid \binom{tL_k P - 1}{k} \quad \text{for every} \quad p \leq k.\]

For \( p \leq l \), (9) holds as before. If \( l < p \leq k \),

\[ p \mid \binom{tL_k P - 1}{k} \]

can only hold if there is a \( j, 1 \leq j \leq k \), for which

\[(10) \quad tL_k P = j \pmod{p^{\alpha_p + 1}}.\]

The number of integers \( t \) with \( 1 \leq t \leq k^2 \), for which (10) holds, is at most
A NEW FUNCTION ASSOCIATED WITH THE PRIME FACTORS OF \((5)\) 649

Thus, by (10) and (11), the number of integers \(t, 1 \leq t \leq k\), for which (10) holds for some prime \(p, l < p \leq k\), is at most

\[
(12) \quad \sum_{t \leq p \leq k} k([k^2/p^2] + 1) < k^3 \sum_{p \geq l} 1/p^2 + k\pi(k).
\]

It easily follows from the prime number theorem that, for \(k > k_0\),

\[
(13) \quad \sum_{p \geq l} 1/p^2 < \frac{2}{l \log l} < \frac{1}{2k}.
\]

From (12) and (13), for \(k > k_0\), the number of integers \(t, 1 \leq t \leq k\), for which (10) holds, is less than \(k^2/2 + k\pi(k) < k^2\). Thus, there is a \(t \leq k^2\) with (9) holding for every \(p \leq k\). Thus, \(g(k) < k^2 L_4 P_1\) as stated. The value 6 could be replaced by a smaller constant, but we cannot prove \(g(k) < L_4\), which seems to hold for all \(k\).

It is well known that \(L_4 < \exp(k(1 + o(1)))\) and \(k^2 P_1 < \exp(o(k))\). Thus, \(g(k) < \exp(k(1 + o(1)))\). So \(g(k) < L_4\) should be achievable.

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