ON ABUNDANT-LIKE NUMBERS

BY

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Problem 188, [3], stated: Apart from finitely many primes $p$ show that if $n_p$ is the smallest abundant number for which $p$ is the smallest prime divisor of $n_p$, then $n_p$ is not squarefree.

Let $2=p_1<p_2<\cdots$ be the sequence of consecutive primes. Denote by $n_k^{(c)}$ the smallest integer for which $p_k$ is the smallest prime divisor of $n_k^{(c)}$ and $\sigma(n_k^{(c)}) \geq cn_k^{(c)}$ where $\sigma(n)$ denotes the sum of divisors of $n$. Van Lint's proof, [3], gives without any essential change that there are only a finite number of squarefree integers which are $n_k^{(c)}$'s for some $c \geq 2$. In fact perhaps 6 is the only such integer. This could no doubt be decided without too much difficulty with a little computation.

Note that $n_k^{(2)}=945=3^3 \cdot 5 \cdot 7$. I will prove that $n_k^{(2)}$ is cubefree for all $k > k_0$, the exceptional cases could easily be enumerated. The cases $1 < c < 2$ causes unexpected difficulties which I have not been able to clear up completely. I will use the methods developed in the paper of Ramanujan on highly composite numbers [1]. A well known result on primes states that for every $s$, [2],

\[
\sum_{p < x} \frac{1}{p} = \log \log x + B + O \left( \frac{1}{(\log x)^s} \right).
\]

(1) implies

\[
\sum_{x < p < x^{1+a}} \frac{1}{p} = \log(1+a) + O \left( \frac{1}{(\log x)^s} \right).
\]

It would be interesting to decide whether

\[
\sum_{x < p < x^{1+a}} \frac{1}{p} - \log(1+a)
\]

changes sign infinitely often. I do not know if this question has been investigated.

**Theorem 1.** $n_k^{(2)}$ is cubefree for all $k > k_0$.

Clearly (see [1])

\[
k_k^{(3)} = \prod_{i=0}^{1} p_k^{a_k}, \quad a_0 \geq a_1 \geq \cdots \geq a_i.
\]

It is easy to see that

\[
\exp \left( \sum_{i=1}^{l} \frac{1}{p_{k+i}^2 - 1} \right) \geq \frac{\sigma(n_k^{(c)})}{n_k^{(c)}} \geq \exp \left( \sum_{i=1}^{l} \frac{1}{p_{k+i}} - \sum_{i=1}^{l} \frac{1}{p_{k+i}^2} \right).
\]

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This, together with the definition of \( n_k^{(e)} \), and a simple computation imply
\[
\sum_{i=1}^{1} \frac{1}{p_{k+i}} = \log c + o\left(\frac{1}{k}\right)
\]
and hence by (2) we have
\[
\lim_{k \to \infty} \frac{p_{k+1}}{p_k^c} = 1.
\]

Let \( c = 2 \). We show that if \( \varepsilon > 0 \) is small enough then for every \( u \) such that \( p_{k+u} < (1 + \varepsilon)p_k \). We have
\[
\alpha_{k+u} \geq 2.
\]

If (6) would be false put
\[
N = n_k^{(2)} p_{k+u} p_{k+u+1} p_{k+u+2} p_{k+1} p_{k+1-1} < n_k^{(2)}
\]
by (5) and \( p_{k+u+1} < 2p_k \). Further for \( k > k_0 \), \( p_{k+u+1} < (1 + 2\varepsilon)p_k \) by the prime number theorem. Thus for sufficiently small \( \varepsilon \) we have by a simple computation
\[
\frac{\sigma(N)}{N} > \frac{\sigma(n_k^{(2)})}{n_k^{(2)}}.
\]

(7) and (8) contradict the definition of \( n_k^{(2)} \) and thus (6) is proved.

Now we prove Theorem 1. Let \( p_{k+u} \) be the greatest prime not exceeding \( (1 + \varepsilon)p_k \). By the prime number theorem
\[
p_{k+u} > \left(1 + \frac{\varepsilon}{2}\right)p_k.
\]

Assume \( \alpha_k \geq 3 \). Put \( N_k = n_k^{(2)} p_{k+1} p_{k+1}^{-1} p_{k+1-1}^{-1} \). By (5), \( N_1 < n_k^{(2)} \) and by a simple computation \( \sigma(N_1)/N_1 > \sigma(n_k^{(2)})/n_k^{(2)} \), which again contradicts the definition of \( n_k^{(e)} \). This proves Theorem 1.

**Theorem 2.** \( n_k^{(3)} = \prod_{i=0}^{u} p_{k+i}^{2} \prod_{i=u+1}^{1} p_{k+i} \) where
\[
\lim_{k \to \infty} \frac{p_{k+1}}{p_k^2} = 1, \quad \lim_{k \to \infty} \frac{p_{k+u}}{p_k} = 2^{1/2}.
\]

The first equation of (9) is (5), the proof of the second is similar to the proof of Theorem 1 and we leave it to the reader.

Henceforth we assume \( 1 < c < 2 \). It seems likely that for every \( c \) there are infinitely many values of \( k \) for which \( n_k^{(e)} \) is squarefree and also there are infinitely many values of \( k \) for which \( n_k^{(e)} \) is not squarefree. I can not prove this. Denote by \( A \) the set of those values \( c \) for which \( n_k^{(e)} \) is infinitely often not squarefree and \( B \) denotes the set of those \( c \)'s for which \( n_k^{(e)} \) is infinitely often squarefree.

**Theorem 3.** \( A, B \) and \( A \cap B \) are everywhere dense in \((1, 2)\).
We only give the proof for the set $A$, for the other two sets the proof is similar. Let $1 \leq u_1 < v_1 \leq 2$. It suffices to show that there is a $c$ in $A$ with $u_1 < c < v_1$. Let $k_1$ be sufficiently large and let $l_1$ be the smallest integer for which

$$
\prod_{i=0}^{l_1} \left( 1 + \frac{1}{p_{k_1+i}} \right) = \sigma \left( \prod_{i=0}^{l_1} p_{k_1+i} \right) / \prod_{i=0}^{l_1} p_{k_1+i} > u_1
$$

Put $x_1 = \prod_{i=0}^{l_1} p_{k_1+i}$. We show that for every $x$ satisfying

$$
u_1 < \frac{\sigma(x_1)}{x_1} < \frac{\sigma(p_{k_1} x_1)}{p_{k_1} x_1} < v_1
$$

we have

$$n^{(a)}_{k_1} = p_{k_1} x_1.
$$

To prove (12) write

$$n^{(a)}_{k_1} = \prod_{i=1}^{j} p_{k_1+i}^{x_i}, \quad \alpha_0 \geq \alpha_1 \geq \ldots \geq \alpha_j.
$$

We show $\alpha_0 = 2, \alpha_1 = 1, j = l_1$ which implies (12). Assume first $\alpha_1 \geq 2$. For sufficiently large $k_1$ we have from (5)

$$T = n^{(a)}_{k_1} p_{k_1+1} p_{k_1}^{x_1} < n^{(a)}_{k_1} \quad \text{and} \quad \frac{\sigma(T)}{T} > \frac{\sigma(n^{(a)}_{k_1})}{n^{(a)}_{k_1}}
$$

which contradicts the definition of $n^{(a)}_{k_1}$. Thus $\alpha_1 = 1, j \leq l_1$ follows from (5) and (11) and $\alpha_0 < 3$ follows like $\alpha_1 = 1$. Thus by (10) $j = l$ and (12) is proved. Thus for the interval (11) $n^{(a)}_{k_1}$ is not squarefree. Now put

$$u_2 = \frac{\sigma(x_1)}{x_1}, \quad v_2 = \frac{\sigma(p_{k_1} x_1)}{p_{k_1} x_1}.
$$

Let $p_{k_2}$ be sufficiently large and repeat the same argument for $(u_2, v_2)$ which we just need for $(u_1, v_1)$. We then obtain $x_2 = \prod_{i=0}^{l_2} p_{k_2+i} x_2$ so that for every $x$ in $u_2 < \sigma(x_2)/x_2 < \alpha < \sigma(p_{k_2} x_2)/p_{k_2} x_2 < v_2$ $n^{(a)}_{k_2} = p_{k_2} x_2$ and is thus not squarefree. This construction can be repeated indefinitely and let $c$ be the unique common point of the intervals $(u_i, v_i), i = 1, 2, \ldots$. Clearly $n^{(a)}_{k_i} = p_{k_i} x_i$ is not squarefree for infinitely many integers $k_i$ or $c$ is in $A$ which completes the proof of Theorem 3.

I can prove that $B$ has measure 1 and that for a certain $\alpha$ every $1 < c < 1 + \alpha$ is in $B$. I can not prove the same for $A$. I do not give these proofs since it seems very likely that every $c, 1 < c < 2$ is in $A \cap B$.

Let $r > 2$ be an integer. It is not difficult to prove by the method used in the proof of Theorem 1 that $p_k^r \mid n^{(r)}_k$ for all $k > k_0(r)$, but for $k > k_0(r), p_{k+1}^{r+1} \mid n^{(r)}_k$ i.e. $n^{(r)}_k$ is divisible by an $r$th power but not an $(r+1)$st power.