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Very Slowly Varying Functions

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Abstract

A real-valued function f of a real variable is said to be φ-slowly varying (φ-s.v.) if $\lim_{x\to\infty} \varphi(x)\,[f(x+\alpha)-f(x)]=0$ for each α. It is said to be uniformly φ-slowly varying (u. φ-s.v.) if $\lim_{x\to\infty} \sup_{\alpha\in I} \varphi(x)\,|f(x+\alpha)-f(x)|=0$ for every bounded interval I.

It is supposed throughout that φ is positive and increasing. It is proved that if φ increases rapidly enough, then every φ-s.v. function f must be u.φ-s.v. and must tend to a limit at ∞. Regardless of the rate of increase of φ, a measurable function f must be u.φ-s.v. if it is φ-s.v. Examples of pairs (φ, f) are given that illustrate the necessity for the requirements on φ and f in these results.

Introduction

The theory of slowly varying functions plays a role in analysis and number theory and has recently come to the fore in probability theory [3]. We consider here some simple, but basic questions about slowly varying functions. We prove four theorems and a lemma.

I. Statement of Results

Let φ be a positive non-decreasing real-valued function defined on $[0, \infty)$ and let f be any real-valued (not necessarily measurable) function defined on $[0, \infty)$. The object of this paper is to study the condition

$$\text{for every } \alpha, \quad \varphi(x)\,[f(x+\alpha)-f(x)] \to 0 \quad \text{as } x\to\infty. \tag{1.1}$$

Whenever (1.1) holds, we will say that f is φ-slowly varying, and abbreviate this by φ-s.v. If (1.1) holds uniformly for α in each bounded interval, then we say that f is uniformly φ-slowly varying (u.φ-s.v.). In other words, f is u.φ-s.v. if

$$\lim_{x\to\infty} \sup_{\alpha\in I} \varphi(x)\,|f(x+\alpha)-f(x)| = 0 \quad \text{for each bounded interval } I.$$

Throughout this paper, the words 'measurable' and 'measure' refer to Lebesgue measure.

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Of course, if f is u.φ-s.v. then it is φ-s.v. The converse is 'almost' true.

THEOREM 1. *If f is φ-slowly varying and measurable, then f is uniformly φ-slowly varying.*

THEOREM 2. *If f is φ-slowly varying and if φ satisfies*

$$\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty, \tag{1.2}$$

then f tends to a finite limit at ∞. Conversely, if

$$\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} = \infty, \tag{1.3}$$

then there is a continuous function f (whose choice depends on φ) that is φ-slowly varying (and, hence, uniformly φ-slowly varying by Theorem 1), but that does not tend to a limit, finite or infinite, at ∞.

THEOREM 3. (a) *If f is φ-slowly varying and if φ satisfies*

$$\varphi(x) \sum_{j=0}^{\infty} \frac{1}{\varphi(x+j)} \leqslant B < \infty \quad \text{for all } x \geqslant 0, \tag{1.4}$$

or, equivalently,

$$\varphi(x) \int_{x}^{\infty} \frac{dt}{\varphi(t)} \leqslant C < \infty \quad \text{for all } x \geqslant 0, \tag{1.4$'$}$$

then f is uniformly φ-slowly varying.

(b) *Conversely, if φ does not satisfy* (1.4), *then there is a function $f = f(\varphi)$ which is φ*-s.v. *but not uniformly φ*-s.v.[3])

The proof of the first part of Theorem 3 may be easily modified to prove the next result.

THEOREM 3$'$. *If f is φ-slowly varying and if ψ is a positive increasing function on $[0, \infty)$ such that φ/ψ is increasing, then f is uniformly φ/ψ-slowly varying provided*

[3]) The completion of this half of the theorem, together with Theorem 4, was inspired by a note communicated to us by Tord Ganelius [5].

that

$$\varphi(x) \sum_{j=0}^{\infty} \frac{1}{\varphi(x+j)} \leqslant B\psi(x) \tag{1.5}$$

for all $x \geqslant 0$ *and some finite constant* B.

The following result shows that the more strongly (1.4) fails, the more disjoint become the conditions of slowly varying and of uniformly slowly varying.

THEOREM 4. *If*

$$\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} = \infty,$$

then there is a function $f = f(\varphi)$ *which is* φ*-slowly varying, but not even uniformly* 1*-slowly varying.*

The changes of variables $h = e^{-x}$, $a = e^{-\alpha}$, $f(x) = g(e^{-x})$, $\eta(h) = 1/\varphi(\log 1/h)$ convert condition (1.1) to

$$\text{for every } a > 0, \quad \frac{g(ah) - g(h)}{\eta(h)} \to 0 \quad \text{as } h \to 0+ \tag{1.6}$$

and conditions (1.2) and (1.4) respectively, to

$$\sum_{n=1}^{\infty} \eta(e^{-n}) < \infty$$

and

$$\frac{1}{\eta(e^{-x})} \sum_{k=0}^{\infty} \eta(e^{-x-k}) \leqslant B < \infty.$$

From (1.6), we see that for studying differentiation theory, the function $\varphi(x) = e^x$, which corresponds to $\eta(h) = h$, is of special import. In fact, Theorem 2 with $\varphi(x) = e^x$ provides a negative answer to question (c) on page 501 of [1]. Another change of variables converts our study to that of multiplicatively slowly oscillating functions – we omit the details (see [6], p. 79). The next lemma supplies an affirmative answer to question (b) on page 501 of [1].

LEMMA 1. *The function f is* φ*-slowly varying if it satisfies the apparently weaker condition*

$$\left.\begin{array}{c} \textit{for each } \lambda \textit{ belonging to a set } E \textit{ of positive measure}, \\ \varphi(x)[f(x+\lambda) - f(x)] \to 0 \quad \textit{as} \quad x \to \infty. \end{array}\right\} \tag{1.7}$$

II. Proofs of Results

Proof of Theorem 1. We give a slight variation on the proof given in [6; pp. 81–82] for the case $\varphi(x)\equiv 1$. We assume that f is measurable and φ-s.v. For simplicity, we will prove that

$$\lim_{x\to\infty}\sup_{\alpha\in[0,1]}\varphi(x)\,|f(x+\alpha)-f(x)|=0. \tag{2.1}$$

Supposing, by way of contradiction, that (2.1) fails, there is a $\delta>0$, and there exist sequences $\{x_n\}$ and $\{\alpha_n\}$ such that $x_n\to\infty$ and $\alpha_n\in[0,1]$ such that for each positive integer n,

$$\varphi(x_n)\,|f(x_n+\alpha_n)-f(x_n)|>\delta. \tag{2.2}$$

Let

$$V_n=\{\alpha\in[0,2]\colon\ |f(\alpha+x_k)-f(x_k)|\,\varphi(x_k)<\delta/2\ \text{ for all }\ k\geqslant n\},$$
$$W_n=\{\beta\in[0,1]\colon\ |f(\beta+\alpha_k+x_k)-f(\alpha_k+x_k)|\,\varphi(x_k+\alpha_k)<\delta/2\ \text{ for all }\ k\geqslant n\}$$

and let

$$W_n'=\alpha_n+W_n=\{\eta\colon\ \eta=\alpha_n+\beta\ \text{ for some }\ \beta\in W_n\}.$$

Since $V_n\subseteq V_{n+1}$ and since every $\alpha\in[0,2]$ lies in some V_n, we have $|V_n|>\frac{3}{2}$ if n is sufficiently large, where $|\cdot|$ denotes Lebesgue measure. Similarly, $|W_n'|=|W_n|>\frac{1}{2}$ if n is sufficiently large. Since $W_n'\subseteq[0,2]$, we see that $W_n'\cap V_n$ is not empty for some large n. This leads to a contradiction, since if $\gamma\in W_n'$, we have

$$\begin{aligned}|f(\gamma+x_n)-f(x_n)|\,\varphi(x_n)\geqslant{}&|f(\alpha_n+x_n)-f(x_n)|\,\varphi(x_n)\\&-\left\{|f(\gamma+x_n)-f(\alpha_n+x_n)|\,\varphi(x_n+\alpha_n)\,\frac{\varphi(x_n)}{\varphi(x_n+\alpha_n)}\right\}\\&>\delta-\delta/2=\delta/2\end{aligned}$$

so that γ cannot belong to V_n.

Proof of Theorem 2. We begin with the proof of the first assertion, and suppose that $\sum 1/\varphi(n)<\infty$. If f satisfies (1.1), then f cannot tend to an infinite limit at ∞, since for every positive integer n,

$$|f(n)|\leqslant|f(1)|+\sum_{k=1}^{n-1}|f(k+1)-f(k)|\leqslant|f(1)|+B\sum_{k=1}^{\infty}1/\varphi(k)<\infty.$$

Therefore, if f does not have a finite limit at ∞, we may assume without loss of generality that

$$\limsup_{x\to\infty}f(x)>1\quad\text{and}\quad\liminf_{x\to\infty}f(x)<-1, \tag{2.3}$$

since we could otherwise replace f by $cf+d$ for suitable constants c and d. Since f is

φ-s.v., we see in particular that

$$\varphi(x)\,|f(x+1)-f(x)|<1 \tag{2.4}$$

if x is sufficiently big, say $x \geqslant M$. Also, since $\sum 1/\varphi(n)$ converges, we have

$$\sum_{n=[x]}^{\infty} \frac{1}{\varphi(n)} < \tfrac{1}{2} \tag{2.5}$$

if $x \geqslant N$, say. By (2.3) we may find two numbers x and y with $x > y$ and

$$x > \max(M, N), \quad f(x) > 1$$

and

$$y > \max(M, N), \quad f(y) < -1 .$$

This leads to a contradiction since on the one hand

$$|f(x+n)-f(y+n)| = \left|\frac{\varphi(y+n)\,[f(y+n+(x-y))-f(y+n)]}{\varphi(y+n)}\right| \leqslant 1$$

for n a sufficiently large positive integer, while on the other hand, for any positive integer n,

$$f(x+n) = f(x) + \sum_{k=1}^{n} [f(x+k)-f(x+k-1)] > f(x) - \sum_{k=1}^{n} \frac{1}{\varphi(x+k-1)}$$

$$\geqslant f(x) - \sum_{k=[x]}^{\infty} \frac{1}{\varphi(k)} > 1 - \tfrac{1}{2} = \tfrac{1}{2},$$

and similarly $f(y+n) < -\frac{1}{2}$, so that $f(x+n) - f(y+n) > 1$.

To prove the second half of Theorem 1, let a non-decreasing positive function φ be given that satisfies (1.3), namely, $\sum 1/\varphi(n) = \infty$. We will construct a continuous function f that is φ-s.v. and that satisfies

$$\limsup_{x\to\infty} f(x) = +\infty, \quad \liminf_{x\to\infty} f(x) = -\infty . \tag{2.6}$$

Let $A = A(\varphi)$ be the set of positive integers m satisfying $\varphi(m+1) \leqslant 2\varphi(m)$. By (1.3), we have

$$\sum_{n\in A} \frac{1}{\varphi(n)} = \sum_{n=1}^{\infty} \frac{1}{\varphi(n)} - \sum_{n\notin A} \frac{1}{\varphi(n)} = \infty \tag{2.7}$$

since

$$\sum_{n\notin A} \frac{1}{\varphi(n)} \leqslant \frac{1}{\varphi(1)} + \frac{1}{2}\,\frac{1}{\varphi(1)} + \frac{1}{2^2}\,\frac{1}{\varphi(1)} + \cdots = \frac{2}{\varphi(1)} < \infty .$$

In particular, A is infinite, and we write $A=\{m_1, m_2, m_3, \ldots\}$. Now there are positive constants a_i with $a_{i+1}<a_i$ for $i=1, 2, 3, \ldots$ and $a_i\varphi(m_i)\to 0$ as $i\to\infty$ and $\sum_{i=1}^{\infty} a_i=\infty$ (see [2], p. 47). We now define a sequence $\{b_i\}$ by $b_i=\pm a_i$, where the signs are chosen in blocks so that $\sum b_i$ has both $+\infty$ and $-\infty$ as limits of subsequences of its partial sums. We define f by $f=0$ on $[0, m_1]$, $f=b_1$ on $[m_1+1, m_2]$, $f=b_1+b_2$ on $[m_2+1, m_3]$, $\ldots, f=b_1+b_2+\cdots+b_k$ on $[m_k+1, m_{k+1}], \ldots$, and extend f to be linear and continuous on each interval $[m_k, m_k+1]$, $k=1, 2, 3, \ldots$. It is clear that (2.6) holds. To verify that f is φ-s.v., we note that

$$\left.\begin{aligned}\varphi(x)\,|f(x+\alpha)-f(x)| = {} & \varphi(x)\,|f(x+\alpha)-f([x+\alpha]+1) \\ & + \sum_{i=0}^{[x+\alpha]-[x]} \{f([x]+i+1)-f([x]+i)\} \\ & + f([x])-f(x)\,| \leqslant \\ & |f([x+\alpha])-f([x+\alpha]+1)\,|\varphi([x+\alpha]+1) \\ & + \sum_{i=0}^{[x+\alpha]-[x]} |f([x]+i+1) \\ & - f([x]+i)\,|\varphi([x]+i+1) \\ & + |f([x])-f([x]+1)\,|\varphi([x]+1),\end{aligned}\right\} \tag{2.8}$$

since f is monotone between consecutive integers and φ is non-decreasing. For fixed α, there are at most $[\alpha]+4$ terms on the right hand side of (2.8), and as $x\to\infty$, each term tends to 0 since

$$|f(m)-f(m+1)\,|\varphi(m+1)=\begin{cases}0 & \text{if} \quad m\notin A \\ a_k\varphi(m_k+1) & \text{if} \quad m=m_k\in A\end{cases}$$

and

$$a_k\varphi(m_k+1)=\frac{\varphi(m_k+1)}{\varphi(m_k)}\,a_k\varphi(m_k)\leqslant 2a_k\varphi(m_k),$$

which tends to 0 as $k\to\infty$.

Proof of Theorem 3(a). We prove a stronger result than asserted, using the same idea we used to prove Theorem 2. Namely, we prove that if f is φ-s.v. and if φ satisfies (1.4), then

$$\lim_{x\to\infty}\sup_{\alpha\geqslant 0}\varphi(x)\,|f(x+\alpha)-f(x)\,|=0. \tag{2.9}$$

Since (1.4) implies (1.2), we know by Theorem 2 that f tends to a finite limit L at ∞. It follows from (2.9), on letting $\alpha\to\infty$, that

$$\lim_{x\to\infty}\varphi(x)\,|f(x)-L|=0. \tag{2.10}$$

For the proof of (2.9), suppose it is false. Then we can find $\delta>0$ and arbitrarily large

x such that for $\alpha=\alpha(x)\geqslant 0$ we have

$$\left.\begin{aligned}|f(x+\alpha+k)-f(x+k)| &= \Bigg|\sum_{j=0}^{k-1}\{f(x+\alpha+j+1)-f(x+\alpha+j)\}\\ &\quad -\sum_{j=0}^{k-1}\{f(x+j+1)-f(x+j)\}+f(x+\alpha)-f(x)\Bigg|\\ &\geqslant \frac{\delta}{\varphi(x)}-\sum_{j=0}^{\infty}\frac{\varepsilon(x+\alpha+j)+\varepsilon(x+j)}{\varphi(x+j)},\end{aligned}\right\} \tag{2.11}$$

where $\varepsilon(y)=\varphi(y)|f(y+1)-f(y)|$, which tends to 0 as $y\to\infty$. Now choose x so large in (2.11) that $\varepsilon(y)<\delta/4B$ for $y\geqslant x$, to get $\varphi(x+k)|f(x+\alpha+k)-f(x+k)|>\delta/2$, which contradicts the hypothesis that f is φ-s.v., since $(x+\alpha+k)-(x+k)=\alpha$, which is independent of k.

Proof of Theorem 3(b). From the geometrically evident identity

$$\int_x^\infty \frac{dt}{\varphi(t)}\leqslant\sum_{j=0}^{\infty}\frac{1}{\varphi(x+j)}\leqslant\frac{1}{\varphi(x)}+\int_x^\infty\frac{dt}{\varphi(t)},$$

it follows that (1.4) and (1.4)′ are equivalent. Assume now that (1.4)′ fails. Let $\{\beta_\lambda\}$ be a Hamel basis for the real numbers, i.e., every real number x has a unique representation $x=\sum_{k=1}^n r_k\beta_{\lambda_k}$ with a finite number $n=n(x)$ of non-zero rationals $\{r_k\}$. Evidently, $|n(x+\alpha)-n(x)|\leqslant n(\alpha)$. One may easily construct a function $\psi\downarrow 0$ such that also

$$\limsup_{x\to\infty}\varphi(x)\int_x^\infty\frac{\psi(t)}{\varphi(t)}\,dt=\infty. \tag{2.12}$$

Let

$$f(x)=\int_0^{x+n(x)-1}\frac{\psi(t)}{\varphi(t)}\,dt.$$

If α is fixed, then, since $\psi/\varphi\downarrow$ and both limits of integration are greater than x,

$$\begin{aligned}\varphi(x)\,|f(x+\alpha)-f(x)| &= \varphi(x)\left|\int_{x+n(x)-1}^{x+\alpha+n(x+\alpha)-1}\frac{\psi(t)}{\varphi(t)}\,dt\right|\\ &\leqslant\varphi(x)\cdot\frac{\psi(x)}{\varphi(x)}\cdot|\alpha+n(x+\alpha)-n(x)|\leqslant\psi(x)(\alpha+n(\alpha)),\end{aligned}$$

which tends to 0 as x tends to infinity so that f is φ-s.v. But f is not uniformly φ-s.v. In fact,

$$\limsup_{x\to\infty}\,(\sup_{\alpha\in[0,1]}\varphi(x)\,|f(x+\alpha)-f(x)|)=\infty .$$

To see this, let $M>0$ be given. Pick $y_0>M$ such that

$$\varphi(y_0)\int_{y_0}^{\infty}\frac{\psi(t)}{\varphi(t)}\,dt>M .$$

Pick $y_1>y_0$ such that $n(y_1)=1$ and so close to y_0 that

$$\varphi(y_1)\int_{y_1}^{\infty}\frac{\psi(t)}{\varphi(t)}\,dt>M$$

also. (This can be done since all the members of the dense set $\{r\beta_{\lambda_1}\colon r \text{ is rational}\}$ satisfy $n=1$.) Finally, pick $\alpha\in[0, 1]$ so that $n(y_1+\alpha)$ is so big that

$$\varphi(y_1)\int_{y_1+n(y_1)-1}^{y_1+\alpha+n(y_1+\alpha)-1}\frac{\psi(t)}{\varphi(t)}\,dt=\varphi(y_1)\,|f(y_1+\alpha)-f(y_1)|$$

is also greater than M. This shows the lim sup to be greater than (an arbitrarily chosen) M and hence infinite.

Proof of Theorem 4. The proof proceeds essentially as the proof of 3(b) above, so we will be brief. Equivalent to our assumption is the equality $\int_x^\infty dt/\varphi(t)=\infty$. Choose $\psi\downarrow 0$ such that $\int_x^\infty \psi(t)/\varphi(t)\,dt=\infty$. Define

$$f(x)=\int_0^{x+n(x)}\frac{\psi(t)}{\varphi(t)}\,dt .$$

For fixed α we have

$$\varphi(x)\,|f(x+\alpha)-f(x)|=\varphi(x)\left|\int_{x+n(x)}^{x+\alpha+n(x+\alpha)}\frac{\psi(t)}{\varphi(t)}\,dt\right|\leqslant\varphi(x)\cdot\frac{\psi(x)}{\varphi(x)}\cdot(\alpha+n(\alpha))$$

which tends to 0; while for each x

$$\sup_{\alpha\in[0,1]}|f(x+\alpha)-f(x)|=\sup_{\alpha\in[0,1]}\left|\int_{x+n(x)}^{x+\alpha+n(x+\alpha)}\frac{\psi(t)}{\varphi(t)}\,dt\right|=\infty$$

since $n(x+\alpha)$ may be arbitrarily large.

Proof of Lemma 1. Assume that (1.7) holds and that $\lambda, \mu \in E$ with $\lambda > \mu$. We must prove that (1.1) holds. First we have the inequality

$$\begin{aligned}\varphi(x)\,|f(x+\lambda-\mu)-f(x)| &= \Big| -\frac{\varphi(x)}{\varphi(x+\lambda-\mu)}\,\varphi(x+\lambda-\mu)\,\{f(x+\lambda)\\ &\quad - f(x+\lambda-\mu)\} + \varphi(x)\,\{f(x+\lambda)-f(x)\}\Big|\\ &\leqslant \varphi(x+\lambda-\mu)\,|f((x+\lambda-\mu)+\mu)-f(x+\lambda-\mu)| + \varphi(x)\,|f(x+\lambda)-f(x)|.\end{aligned}$$

Then we apply Steinhaus' Theorem (see [4; p. 68] or [8; pp. 97–99]) that the difference set of a set of positive measure contains an open interval that contains 0, to deduce that (1.1) holds for all sufficiently small α. Now repeated application of the inequality

$$\varphi(x)\,|f(x+2\alpha)-f(x)| \leqslant \varphi(x+\alpha)\,|f(x+2\alpha)-f(x+\alpha)| + \varphi(x)\,|f(x+\alpha)-f(x)|$$

completes the proof. (See also [7; pp. 266–267], and [1; p. 493].)

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