COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
10. INFINITE AND FINITE SETS, KESZTHELY (HUNGARY), 1973.

ANTI-RAMSEY THEOREMS

P. ERDŐS – M. SIMONOVITS – V.T. SÓS

0. NOTATIONS

Let H be a fixed graph, n a given integer. For some m it is possible to colour the edges of K^n with m colours so that no subgraph of K^n each edge of which has different colour will be isomorphic to H. The maximum m will be denoted by $f(n, H)$. The present paper investigates the dependence of $f(n, H)$ on n and H.

A graph considered below will never contain loops or multiple edges. If G is a graph, $e(G), v(G)$ and $k(G)$ will denote the number of edges, vertices and the chromatic number respectively. $E(G), V(G)$ will denote the edge – and vertex-set of G. If G_1 and G_2 are subgraphs of G, $e(G_1, G_2)$ and $E(G_1, G_2)$ will denote the number of edges and the set of edges joining G_1 and G_2 in G, respectively. If G^m denotes a graph, m (the superscript) will always denote the number of vertices of it. Thus e.g., P^k and C^k denote the path and cycle of k vertices. The set of neighbours of a vertex $x \in G$ will be denoted by $N(x)$, the cardinality of $N(x)$, that is, the degree of x is denoted by $d(x)$. If E is a set, $|E|$ denotes its cardinality.

$K_d(n_1, \ldots, n_d)$ denotes the complete d-partite graph with n_i vertices in its i-th class. $K^d =: K_d(1, \ldots, 1)$ is the complete d-graph. If G is a graph and A is either a set of vertices and edges of G or a subgraph of G, $G - A$ denotes the graph obtained by omitting the edges and vertices of A and also all the edges having endvertices among the omitted ones. If A is a set of edges of the complementary graph of G, $G + A$ denotes the graph obtained by adding the edges of A to G. $\bar{G}$ denotes the complementary graph of G.

1. INTRODUCTION

If K^n is edge-coloured in a given way and a subgraph U contains no two edges of the same colour, then U will be called a totally multicoloured subgraph of K^n and we shall say that K^n contains a TMC U. Let $f(n, H)$ be the maximum number of colours K^n can be coloured with without containing a TMC H. The problem of determining $f(n, H)$ is connected not so much to Ramsey-theorem than to Turán-type problems. For a given family $\mathscr{H}$ of finite graphs

$$\operatorname{ext}(n, \mathscr{H}) =: \max\{e(G^n): H \not\subset G^n \quad \text{if} \quad H \in \mathscr{H}\},$$

that is, let $\operatorname{ext}(n, \mathscr{H})$ be the maximum number of edges a graph G^n can have if it has no subgraph from $\mathscr{H}$. The graphs attaining the maximum for a given n are called extremal graphs.

The main result of [1], asserts that if $d + 1 = \min_{H \in \mathscr{H}} k(H)$, then

$$\operatorname{ext}(n, \mathscr{H}) \Big/ \binom{n}{2} \to \left(1 - \frac{1}{d}\right) \quad \text{when} \quad n \to \infty. \tag{1}$$

The corresponding result for $f(n, H)$ is formulated in Theorem 1:

Theorem 1. *Let*

$$d + 1 = \min\{k(H - e): e \in E(H)\}. \tag{2}$$

Then

$$\frac{f(n, H)}{\binom{n}{2}} \to \left(1 - \frac{1}{d}\right) \quad \text{if} \quad n \to \infty. \tag{3}$$

A similar result holds for uniform hypergraphs. Instead of defining the hypergraphs we refer to [2], [3] and for the sake of simplicity we restrict ourselves to the case of 3-uniform hypergraphs in the proof. Let $\mathrm{ext}_k(n, \mathscr{H})$ denote the maximum number of k-tuples a k-uniform hypergraph of n vertices can have without containing a subhypergraph from $\mathscr{H}$. Let $f_k(n, H)$ denote the maximum number of colours a complete k-uniform n-graph $K^n_{(k)}$ can be coloured with if it does not contain a TMC copy of the k-uniform hypergraph H. One can easily show [4] that $\mathrm{ext}_k(n, H) \Big/ \binom{n}{k}$ is convergent when $n \to \infty$.

Theorem 2. *Let H be a k-uniform hypergraph and let $\mathscr{H} = \{H - e$: e is a k-tuple of $H\}$. Then*

$$f_k(n, H) - \mathrm{ext}_k(n, \mathscr{H}) = o(n^k)\,. \tag{4}$$

(In other words, $f_k(n, H) \Big/ \binom{n}{k}$ and $\mathrm{ext}_k(n, \mathscr{H}) \Big/ \binom{n}{k}$ converge to the same limit.)

One can also ask, what is the structure of the extremal colourings i.e., how is K^n coloured when it is coloured by $f(n, H)$ colours and it does not contain TMC H. Using the results of [5] and [6] instead of (1) and combining the method of the proofs of Theorems 1, 2 with that of Theorem 4 one can prove the following assertion:

Theorem 3. *Let us consider a K^n coloured by $f(n, H)$ colours and not containing* TMC *H. One can subdivide $V(K^n)$ into d sets $A_1, \dots, A_d$ (where d was defined in Theorem 1) so that all but $o(n^2)$ edges joining different classes have own colours (i.e., colours used only once) and all the edges joining vertices of the same class A_j have $o(n^2)$ colours altogether,* $(j = 1, 2, \dots, d)$.

The proof of this theorem will not be published here.

The theorems above were the general ones. Now we turn to theorems concerning special choices of H.

Theorem 4. *Let $p \geq 4$. There exists an n_p such that if $n > n_p$, then*

(5) $\qquad f(n, K^p) = \text{ext}\,(n, K^{p-1}) + 1\,.$

Further, if K^n *is coloured by* $f(n, K^p)$ *colours and it contains no* TMC K^n, *then its colouring is uniquely determined: one can divide the vertices of* K^n *into* d *classes* $A_1, \ldots, A_d$ *so that each edge joining vertices from different* A_i*'s has its own colour (that is, a colour used only once) and each edge of form* (x, y) *where* x *and* y *belong to the same* A_i *has the same colour, independent from* x, y *and* i.

Remark 1. The second part of Theorem 4 means that an extremal colouring of K^n can be obtained from an extremal graph S^n for ext (n, K^{p-1}) by colouring the edges of S^n differently and the edges of $\bar{S}^n$ by one extra colour. A theorem of G. Dirac asserts that ext $(n, K^{p-1}) =$ $=$ ext $(n, K^p - e)$, moreover, the extremal graph is the same if $n \geqslant 2p$. Dirac's theorem, and Theorems 1, 2 throw light on the background of Theorem 4.

Remark 2. If $d = 1$, the information yielded by Theorem 1 is that $f(n, H) = o(n^2)$. Unlike in case $d \geqslant 2$ we do not get upper and lower bounds the ratio of which tends to 1. This case will be called degenerated. We do not have too much result in this case, mainly, because the corresponding degenerated problems of Turán-type are unsolved and seem pretty difficult.

Two degenerated problems will be discussed here: the problems of C^k and P^k.

Conjecture 1.

$$f(n, C^k) = n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) + O(1)\,.$$

The conjecture says that the best way to colour K^n so that no TMC C^k would occur is to divide the points into $\frac{n}{k-1}$ groups of $k-1$ vertices and then colour all the edges joining vertices of the same group by different colours, and the edges joining vertices from different groups colour by $\frac{n}{k-1}$ further colours in the following way: the vertices of the i-th group are joined by the i-th extra colour to the vertices of the j-th group if $j > i$. We do not assert, however the uniqueness of the extremal colourings. This conjecture will be proved only for $k = 3$ in Theorem 5.

Conjecture 2. Let t be a given integer, $\epsilon = 0, 1$ and $k = 2t + 3 + \epsilon$. Then

$$f(n, P^k) = tn - \binom{t+1}{2} + 1 + \epsilon \quad \text{if} \quad n \geqslant \frac{5t + 3 + 4\epsilon}{2} \tag{6}$$

and

$$f(n, P^k) = \binom{k-2}{2} + 1 \quad \text{if} \quad k \leqslant n \leqslant \frac{5t + 3 + 4\epsilon}{2}. \tag{7}$$

Further, the only extremal colourings corresponding to (6) are the following ones: t vertices $x_1, \dots, x_t \in K^n$ can be choosen so that all the edges of form (x_j, y), $j = 1, \dots, t$, $y \in K^n$, have different colours and the edges of $K^n - \{x_1, \dots, x_t\}$ are coloured by one or two (more exactly, by $1 + \epsilon$) further colours. The only extremal colourings corresponding to (7) are the following ones: $k - 2$ vertices $x_1, \dots, x_{k-2}$ can be chosen in K^n so that all the edges (x_i, x_j) have different colours and all the other edges have the same extra colour.

Remark 3. If t is odd, for $n = \dfrac{5t + 3 + 4\epsilon}{2}$ we have two different extremal colourings in the conjecture.

We can prove the following theorem:

Theorem 5. *There exists a constant* c *such that if* $n \geqslant \dfrac{5t + 3 + c}{2}$ *then Conjecture* 2 *is valid.*

It is not too difficult to improve some estimates of our proof on Theorem 5 and get

Theorem 6. *If* t *is sufficiently large, then Conjecture* 2 *is valid.*

However, even the proof of Theorem 5 is rather long and we cannot prove Theorem 6 in a satisfactorily short way. The proofs of Theorems 5, 6, will be published later.

2. PROOFS OF THEOREMS 1, 2

Let $\mathscr{L}$ be a given family of graphs. Let $\mathscr{L}^-$ denote the family of graphs $H - e$ where H is from $\mathscr{L}$ and e is an edge of it. Let $\mathscr{L}^+$ be the family of graphs G having the following property:

If we colour the edges of G by $e(G)$ different colours and colour $\bar{G}$ in an arbitrary way, then the obtained colouring of $K^{v(G)}$ always contains a TMC H from $\mathscr{L}$.

Here $\mathscr{L} = \{H\}$ will be assumed.

Lemma 1. *If* $\mathscr{L}^* \subseteq \mathscr{L}^+$, *then*

$$(8) \qquad 1 + \mathrm{ext}\,(n, \mathscr{L}^-) \leqslant f(n, H) \leqslant \mathrm{ext}\,(n, \mathscr{L}^*) .$$

Proof. Let G^n be an extremal graph for $\mathscr{L}^-$. Let us colour the edges of G^n by different colours "1", "2", . . . , "m", where $m = \mathrm{ext}\,(n, \mathscr{L}^-) = e(G^n)$ and let us colour the edges of $\bar{G}^n$ by "0". The obtained colouring of K^n contains no TMC H and this implies the left side of (8).

Let us call G^n a representation of a colouring of K^n if it contains exactly one edge of each colour. Let us consider an extremal colouring of K^n and a representation G^n of it. By definition of $\mathscr{L}^+$, G^n contains no subgraph from $\mathscr{L}^*$. Hence

$$f(n, H) = e(G^n) \leqslant \mathrm{ext}\,(n, \mathscr{L}^*) . \qquad \text{Q.E.D.}$$

Remark 4. One can easily prove that

$$(9) \qquad f(n, H) = \mathrm{ext}\,(n, \mathscr{L}^+) .$$

However, since $\mathscr{L}^+$ is an infinite family of graphs which generally is difficult to describe we have trouble in utilizing the whole amount of information in (9).

Remark 5. In our proofs we shall use the following observation: Let $G_1, G_2 \in \mathscr{L}^+$. If we omit an edge (x, x') from G_1 and an edge (y, y') from G_2 and identify x with y and x' with y', then the obtained graph $G_3 \in \mathscr{L}^+$. Indeed, if we colour G_3 by $e(G_3)$ colours and the complementary graph arbitrarily, either $G_2 \subseteq G_3 + (x, x')$ or $G_1 \subseteq$ $\subseteq G_3 + (x, x')$ is a TMC graph and therefore either $K^{\nu(G_1)}$ or $K^{\nu(G_2)}$ (spanned by G_1 and G_2 respectively) must contain a TMC H.

Proof of Theorem 1. By Remark 5, if we omit an edge $e = (x, x')$ from H so that $k(H - e) = d + 1$ and then take two copies of this $H - e$, say, U_1 and U_2, then by identifying the vertices corresponding to $x \in H - e$ and the vertices corresponding to $x' \in H - e$, we get a $U_3 \in \mathscr{L}^+$. Clearly, $k(U_3) = d + 1$ hence, by (1) and Lemma 1

$$f(n, H) \leqslant \text{ext}\,(n, U_3) = \left(1 - \frac{1}{d} + o(1)\right)\binom{n}{2}. \tag{10}$$

On the other hand, by Lemma 1 and since for any $L \in \mathscr{L}^-$, $k(L) =$ $= k(H - e) \geqslant d + 1$,

$$f(n, H) \geqslant \text{ext}\,(n, \mathscr{L}^-) \geqslant \left(1 - \frac{1}{d} + o(1)\right)\binom{n}{2}. \tag{10^+}$$

This completes our proof.

Proof of Theorem 2. One can immediately see that Lemma 1 and Remark 5 and their proofs remain valid for k-uniform graphs as well. What we cannot directly generalize to hypergraphs is (10) and (10^+). To avoid the clumsy notations we restrict our consideration to the case $k = 3$.

Let G be a 3-uniform hypergraph and let $x_1, \ldots, x_m$ be its vertices, T be the set of its triples. Let $G(t)$ denote the hypergraph obtained from G by replacing x_i by t vertices $x_{i,s}$, $(i = 1, \ldots, m, s = 1, \ldots, t)$. Let $(x_{i_1,s_1}, x_{i_2,s_2}, x_{i_3,s_3})$ be a triple of $G(t)$ if i_1, i_2, i_3 are all different and $(x_{i_1}, x_{i_2}, x_{i_3})$ is a triple of G.

Then a result of P. Erdős [3] generalizing in some sense (1) asserts that

$$\text{ext}_3\,(n, G(t)) - \text{ext}_3\,(n, G) = o(n^3),$$

or, in other words:

$$\lim_{n \to \infty} \frac{\text{ext}_3\,(n, G(t))}{\binom{n}{3}}$$

does exist and *is independent of* t.

This result can easily be generalized as follows:

Let $\mathscr{L}(t) =: \{U(t)\colon\ U \in \mathscr{L}\}$. Then

$$\mathrm{ext}_3\,(n;\ \mathscr{L}(t)) - \mathrm{ext}_3\,(n,\ \mathscr{L}) = o(n^3)\,.$$

If we knew that for any $U = H - e$ (where e is a triple of H), $U(2) \in \in \mathscr{H}^+$, then we were home:

By the generalized Lemma 1 ($\mathscr{L} = \{H\}$, $\mathscr{L}^- = \mathscr{H}$, $\mathscr{L}^+ \supseteq \mathscr{H}(2)$)

$$\mathrm{ext}_3\,(n,\ \mathscr{H}) \leqslant f_3(n, H) \leqslant \mathrm{ext}_3\,(n,\ \mathscr{H}(2)) \leqslant \mathrm{ext}_3\,(n,\ \mathscr{H}) + o(n^3)\,.$$

which implies Theorem 2. But $U(2) \in \mathscr{H}^+$ follows from the corresponding generalization of Remark 5: for a given U there exists a triple (x, y, z) in $\bar{U}$ such that $U + (x, y, z) = H$. Clearly, $U(2)$ contains 3 vertices x', y' and z' and two subgraphs U_1 and U_2 isomorphic to U such that $U_i + (x', y', z') = H_i$ is isomorphic to H $(i = 1, 2)$ and H_1 and H_2 have no common vertices but x', y', z'. Now, colouring the triples of $U(2)$ by different colours and (x', y', z') arbitrarily, by symmetry we may assume that U_1 does not contain the colour of (x', y', z') and therefore H_1 is totally multicoloured.

Q.E.D.

3. PROOF OF THEOREM 4

We shall need the following theorem:

Theorem 6. *Let* $r, d,$ *and* $k \leqslant r/2$ *be given positive integers. Let* $\mathscr{T}$ *be the class of graphs obtainable from* $K_d(r, \ldots, r)$ *by adding* k *edges. Then*

$$\mathrm{ext}\,(n,\ \mathscr{T}) = \mathrm{ext}\,(n,\ K_{d+1}) + k - 1 \quad \textit{if} \quad n \geqslant n_0(r, d, k)\,.$$

Further, S^n *is an extremal graph for* $\mathscr{T}$ *iff one can obtain* S^n *by adding* $k - 1$ *edges to* $K_d(n_1, \ldots, n_d)$, *where* n_i*'s are defined by*

$$(11) \qquad \sum n_i = n \ \textit{and} \ \left|n_i - \frac{n}{d}\right| \leqslant 1, \qquad i = 1, 2, \ldots, d\,.$$

We shall not prove this theorem here, though it follows fairly easily from the much deeper theorems of [7] or it can be proved easily in the same way as its special case $k = 1$ was proved in [6] (Theorem 1*). Theorem 6 will be used here with $k = 2$.

Proof of Theorem 4. Let $k = 2$, $r = 5$ and $d = p - 2$ in Theorem 6. Then $\mathscr{T}$ contains 3 non-isomorphic graphs and one can easily check that all they belong to $\mathscr{L}^+$: $\mathscr{T} \subseteq \mathscr{L}^+$. Therefore, by Lemma 1 and Theorem 6 and since $\mathscr{L}^- = \{K_p - e\}$ and $K_p - e \supseteq K_{p-1}$, we get that

$$\begin{aligned} &1 + \operatorname{ext}(n, K_{p-1}) \leqslant 1 + \operatorname{ext}(n, K_p - e) \leqslant f(n, K_p) \leqslant \\ &\qquad \leqslant \operatorname{ext}(n, \mathscr{T}) = \operatorname{ext}(n, K_{p-1}) + 1 \, . \end{aligned} \tag{12}$$

This gives the "quantitative" part of Theorem 4: the assertion (4). We wish to describe the structure of extremal colouring as well. Let us consider an arbitrary extremal colouring of K^n. Let G^n be a representation of it. Since overall in (12) we have equality, according to the proof of the upper bound in Lemma 1 G^n must be an extremal graph for $\mathscr{T}$. Hence G^n can be obtained from a $K_d(n_1, \ldots, n_d)$ (satisfying (11)) by adding an edge to it.

What we have to show is that if e is the edge of G^n for which $G^n = K_d(n_1, \ldots, n_d) + e$ and f is an arbitrary edge of $\overline{K_d(n_1, \ldots, n_d)}$, then e and f have the same colour. There exists exactly one $f^* \in G^n$ having the same colour as f. Thus $G^n - f^* + f$ is again a representation of this extremal colouring. Therefore there exists an e^* for which

$$(G^n - f^* + f) - e^* \cong K_d(n_1, \ldots, n_d) \, .$$

This implies that $f^* = e$ (unless n is very small), i.e. f and e have the same colour.

5. APPENDIX

A. Here we prove Conjecture 2 for $k = 3$. In fact, it is trivial: Let G^n be a representation of a colouring of K^n by n colours. G^n contains a cycle. Let C^s be the shortest TMC cycle in K^n. If its vertices are $a_1, \ldots, a_s$ in their cyclic order and $s \neq 3$, then the colour of (a_1, a_3) can occur at most in one of the two paths (a_1, a_2, a_3) and $(a_3, a_4, \ldots, a_s)$. In any case we obtain a shorter TMC cycle. Hence $C^s = C^3$. This proves

$$f(n, C^3) = n - 1 ,$$

since $f(n, C^3) \geqslant n - 1$ immediately follows from the following construction: the vertices $x_1, \ldots, x_n \in K^n$ are labelled somehow and we colour (x_i, x_j) by "j" if $i < j$. Thus we get a colouring of K^n by $n - 1$ colours and K^n does not contain any TMC cycle.

B. We could not settle the problem of determining $f(n, K_2(3, 3))$. We do not even know, whether it is greater than $\operatorname{ext}(n, \mathscr{L}^-) = cn^{3/2}$.

C. Let U^4 be the 3-uniform hypergraph on 4 vertices with 3 triples. It is trivial, that

$$f_3(n, U^4) \geqslant \operatorname{ext}_3 (n, \mathscr{L}^-_{U^n}) = \frac{n^2}{6} + o(n^2) .$$

On the other hand, it is easy to see, that

$$f_3(n, U^4) \leqslant \frac{n^2}{4} + O(n) .$$

Probably the lower bound is sharp, but we could not prove it.

REFERENCES

[1] P. Erdős,– M. Simonovits, A limit theorem in graph theory, *Studia, Sci. Math. Hung.*, 1 (1966), 51-57.

[2] C. Berge, *Graphs and Hypergraphs,* North-Holland Publishing Company, 1973.

[3] P. Erdős, On some extremal problems on *r*-graphs, *Discrete Math.*, 1 (1971), 1-6.

[4] Gy. Katona – T. Nemetz – M. Simonovits, On a graph problem of P. Turán, (in Hungarian), *Math. Lapok,* 15 (1964), 228-268.

[5] P. Erdős, Some recent result in extremal graph problems in graph theory, (Results) *Theorey of Graphs, Proc. Symp. Rome,* (1966), 118-123.

[6] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, *Theory of Graphs, Proc. Symp. held at Tihany,* Hungary (1966), Acad. Press, New York, (1968), 279-319.

[7] M. Simonovits, Extremal graph problems with symmetrical extremal graphs, . . . , *Discrete Math.*, 7 (1974), 349-376.

[8] P. Erdős – T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hung.*, 10 (1959), 337-356.