Reprinted from:
APPROXIMATION THEORY, II

ACADEMIC PRESS, INC.
New York San Francisco London

EXTREMAL PROBLEMS ON POLYNOMIALS

P. Erdős

The subject has very many aspects and ramifications and I am not at all competent to deal with many of them and in fact in this short article I do not even have the space. Thus I will eventually concentrate on problems which my collaborators and I investigated. I will try to review them as systematically as possible and will also state a few new problems.

Herzog, Piranian and I in a paper [7] (we will refer to this paper as (I)) investigated several geometric extremal problems for polynomials; we stated there many conjectures some of which were settled (positively or negatively) by Pommerenke and others. First I give a short review of the fate of the conjecture stated in (I). Pommerenke's two papers in which he deals with the problems in (I) are Pommerenke [12] and [14]; we will refer to them as (PI) and (PII), respectively. Some of these problems are discussed by Pommerenke also in [13].

1. Let $-1 \le x_1 \le \dots \le x_n \le 1$. We conjectured and Elbert [4] proved that the measure of the set of real numbers x for which $|\Pi_{i=1}^{n}(x - x_i)| \le 1$ is not greater than $2\sqrt{2}$. The polynomial $1 - x^2$ shows that this is best possible. The proof of Elbert is surprisingly complicated and a simpler one would be desirable.

2. Put $f_n(z) = \Pi_{i=1}^{n}(z - z_i)$. Denote by $E_n(f)$ the set of points where $|f_n(z)| \le 1$. Pólya proved that the area $A_n(E)$ of $E_n(f)$ is $\le \pi$, with equality only for z^n and he also proved that the projection of $E_n(f)$ on every line has

measure < 4. Denote $\varepsilon_n = \min A_n(E)$ where the minimum is to be taken over all $f_n(z)$ where roots are all in $|z| \leq 1$. Using a deep theorem of G. MacLane we proved $\varepsilon_n \to 0$. We conjectured that $\varepsilon_n > n^{-\eta}$ for every $\eta > 0$ and $n > n_o(\eta)$, we also conjectured that one can place a circle of radius Cn^{-1} is $A_n(E)$. Both these conjectures are still open, but Pommerenke proved some weaker results. More generally in (I) we conjectured that if T is any set of transfinite diameter 1 then to every $\varepsilon > 0$, for $n > n_o(\varepsilon)$ there is a polynomial $f_n(z) = \Pi_{i=1}^{n}(z - z_i)$ with $z_i \in T$, so that $A_n(E) < \varepsilon$. This is easy if T is $(-2,+2)$ but the general case is open.

Netaniahu and I [8] proved that there is a constant ρ_c depending on c alone so that if T is a connected set of transfinite diameter $1-c$ then $E_n(f)$ contains a circle of radius ρ_c. Our proof is a pure existence proof and does not give an explicit bound for ρ_c. It would be interesting to determine (or estimate) the best possible value of ρ_c and also the best possible value of $A_n(E)$. The following special case might be of some interest. Let the roots of $f_n(z)$ be in $|z| \leq 1 - c$. Then $E_n(f)$ contains the circle $|z| \leq c$ and thus has area $\geq \pi c^2$. I wonder if this is best possible: Is it true that to every $\varepsilon > 0$ there is an $f_n(z)$ with $|z_i| \leq 1 - c$ and $A_n(E) < \pi c^2 + \varepsilon$?

We proved in (I) that there is a polynomial $f_n(z)$, $|z_i| \leq 1$, for which $E_n(f)$ has $n - 1$ components and that it never can have n components. As far as we know the following question has not been investigated: Let T be a set of transfinite diameter 1. Is it true that for each n there is a polynomial $f_n(z)$ all whose roots are in T and for which $E(f)$ has $\geq n - \gamma_n$ components where γ_n is "small" (certainly $o(n)$ and perhaps bounded; if T is $(-2,+2)$, then $\gamma_n = 0$)? If T has transfinite diameter $1 - c$

then in general $\gamma_n > f(c)n$, but as far as we know also this question has not been investigated.

3. A pretty theorem of H. Cartan states that for a polynomial $f_n(z) = \Pi_1^n(z - z_i)$, $i = 1,\ldots,n$, the lemniscate $E_n(f)$ can always be covered by circles the sum of whose radii is less than 2e. We conjectured in (I) that the correct value here should be 2 (clearly best possible if true). We also conjectured that if $E_n(f)$ is connected then it is contained in a circle of radius 2 and center $\frac{1}{n}\Sigma_{i=1}^n z_i$. This conjecture was proved by Pommerenke (PI).

Two conjectures of (I) which seem to me the most attractive are as follows: The sum of the diameters of the components of $E_n(f)$ is $\leq n2^{1/n}$ (equality for $z^n - 1$) and the length of the curve $|f_n(z)| = 1$ is maximal for $f_n(z) = z^n - 1$. As far as we know no real progress has been made with these conjectures though Pommerenke (PII) proved that the length of $|f_n(z)| = 1$ is less than $74n^2$--whereas the "truth" should be $2n + \mathcal{O}(1)$.

In (I) we made the ill fated conjecture that the number of components of $E_n(f)$ which have a diameter $> 1 + c$ is less than δ_c, δ_c bounded. Pommerenke (PII) showed that nothing could be farther from the truth, in fact he showed that to every ε and k there is an $E_n(f)$ which has more than k components of diameter $> 4 - \varepsilon$. Our conjecture can probably be saved as follows: Denote by $\phi_n(c)$ the largest number of components of diameter $> 1 + c$ which $E_n(f)$ can have. Surely for every $c > 0$, $\phi_n(c) = o(n)$ and hopefully $\phi_n(c) = o(n^{\varepsilon})$. I have no guess about a lower bound for $\phi_n(c)$ also I am not sure whether the growth of $\phi_n(c)$, $1 < c < 4$, depends on c very much.

In (I) we conjectured that if $E_n(f)$ is connected its circumference is $\geq 2\pi$, equality only for z^n. In (PI)

Pommeneke proved this. As far as I know the following question has not yet been investigated. Assume that $E_n(f)$ is convex; how large can its circumference be? Perhaps 8 is the correct bound.

Assume that $E_n(f)$ is connected, let d be its diameter, let b be its width. We conjectured that $b \leq 2$ but in (PII) Pommerenke showed that $b > 2.386$ is possible, he further showed that $b < 2.920$ and that to every $E_n(f)$ there is a direction so that the projection of $E_n(f)$ on this direction has measure < 3.30.

We conjectured that if $E_n(f)$ is connected then to every z_o on the boundary of $E_n(f)$ there is a $z \varepsilon E_n(f)$ with $|z - z_o| \geq 2$. Pommerenke in (PII) disproves this and shows that 2 can be replaced by $(3/4)\sqrt{3}$ which is best possible.

Let $|z_i - z_j| \leq 2$, $1 \leq i < j \leq n$. We conjectured that $\Pi_{1 \leq i < j \leq n} |z_i - z_j|$ assumes its maximum if the z_i are the vertices of a regular polygon of diameter 1. Danzer and Pommerenke [3] disproved this conjecture for even n but it probably holds for odd n. As far as I know it is open for $n \geq 5$.

Assume that $E_n(f)$ has n components. Is it true that its area is maximal if $f_n(z) = z^n - 1$? If true this probably will not be easy to prove. On the other hand the following conjecture (if true) would be probably not difficult to establish. Let $|z_i| = 1$, $1 \leq i \leq n$. Then there is a path joining the origin to the circle $|z| = 2^{1/n}$ on which $|\Pi_{i=1}^n (z - z_i)| \leq 1$.

4. Many of these problems can be extended to higher dimensional spaces or metric spaces. Let us here restrict ourselves to three dimensional Euclidean space with distance $d(z,z')$. If $z_1,\ldots,z_n$ are n points in space, denote by

$E_n(z_i)$ the set of z's for which $\Pi_{i-1}^n d(z,z_i) \leq 1$. Is it true that if $E_n(z_i)$ is connected it is in the interior of the sphere of radius 2 and center $\frac{1}{n}\Sigma_{i=1}^n z_i$? What is the maximum of the volume of $E_n(z_i)^2$? I first thought that the maximum will be attained for the unit sphere , i.e., in case when all the points z_i coincide, but Piranian showed that this is false already for $n = 2$. It would be very interesting to determine the maximal volume of $E_n(z_i)$ and the distribution of the z_i which achieves this maximum.

Many of these problems are of great geometric interest. In the plane the regular polygon usually gives the solution, but nothing corresponds to this in three dimensions. Here is a typical example: A well known theorem of Pólya states that if $|z_i| \leq 1$ then $|\Pi_{1\leq i<j\leq n}(z_i - z_j)|$ is maximal if the z_i are the vertices of the regular n-gon. Assume now that the z_i are in the unit sphere; I can not even guess for which set of n points z_i, $1 \leq i \leq n$ is the maximum of $|\Pi_{1\leq i<j\leq n} d(z,z_i)|$ assumed. Similarly I can not guess for which set z_i is the area of $\Pi_{1\leq i<j\leq n} d(z_i,z_j) = 1$ maximal? (the z_i are unrestricted here).

5. Let $f_n(z) = \Pi_{i=1}^n (z - z_i)$ and assume that $E(f)$ is connected. I conjectured that $\max_{z\varepsilon E_n(f)} |f_n'(z)| < n^2/2$. Pommerenke proved this with $en^2/2$ instead of $n^2/2$. Let α_i be the distance of z_i to the boundaries of the lemniscate. Perhaps $\Sigma_{i=1}^n \alpha_i \geq n(2^{1/n} - 1)$; equality for $z^n - 1$. $\Sigma_{i=1}^n \alpha_i > c$ could remain true if instead of connectedness we assume $|z_i| \leq 1$.

As far as I remember we never considered the following questions which are perhaps not uninteresting. Let $E_n(f)$ be connected, when is its area minimal? Probably when the roots are all real and $\max|f(z)| = 1$ between any two consecutive roots. A related question: Let $E_n(f)$ be connected.

When is $\int_{E_n(f)} |f_n(z)|$ (area integral) maximal and minimal? The maximum is probably achieved for z^n and the minimum when the roots are all real and between any two roots $|f_n(z)|$ assumes the value 1.

In some cases connectedness of $E_n(f)$ can perhaps be replaced by the following condition: No line separates the components of $E_n(f)$. For example: Is it true that $E_n(f)$ is contained in a circle of radius 2 and center $\frac{1}{n}\Sigma_{i=1}^n z_i$ under this assumption? For this section compare [9].

6. I conjectured 35 years ago that if $f_n(\theta)$ is a trigonometric polynomial whose maximum is 1 and all whose roots are real then

$$\int_0^{2\pi} |f_n(\theta)| \leq 4.$$

Let the rational polynomial $f_n(x)$ have all its roots in $(-1,+1)$, $\max_{-1\leq x\leq 1} |f_n(x)| = 1$ and let x_i, x_{i+1} be two consecutive roots of $f_n(x)$. Then

$$\int_{x_i}^{x_{i+1}} |f_n(x)| \leq d_n(x_{i+1} - x_i)$$

where

$$\frac{1}{y_{i+1} - y_i} \int_{y_i}^{y_{i+1}} T_n(y) = d_n,$$

T_n is the Tchebicheff polynomial and y_i, y_{i+1} are two consecutive roots of T_n.

These conjectures and more have all been proved recently by Kristiansen [11] and Saff and Sheil-Small [15].

I proved [5] that the arc length from 0 to 2π of a trigonometric polynomial f_n of degree n satisfying $|f_n(\theta)| \leq 1$ is maximal for $\cos n\theta$. Let $0 < a < b < 2\pi$, is it still true that the variation and arc-length in (a,b) is maximal for $\cos(n\theta + \alpha)$ for a suitable α? Also if $|f_n(x)| \le 1$, $-1 \le x \le 1$ is a rational polynomial of degree n, is it true that $T_n(x)$ has the greatest arc-length? The answer clearly must be yes--only a proof is needed.

7. I would like to call attention to two old problems of mine which perhaps belong more to number theory. Let z_n, $n = 1, 2, \ldots, |z_n| = 1$ be an infinite sequence. Put

$$A_n = \max_{|z|=1} \prod_{i=1}^{n} |z - z_i| .$$

Prove (or disprove) $\overline{\lim}\, A_n = \infty$. This problem is probably difficult. Let B_k be the least upper bound of the numbers

$$\left| \sum_{i=1}^{m} z_i^k \right|, \quad m = 1,2,\ldots .$$

It is easy to see that a sequence $\{z_j\}$, $j = 1,2,\ldots$ exist for which $B_k < Ck$. I conjecture $\overline{\lim}(B_k/k) > 0$. Clunie [2] proved $\overline{\lim}(B_k/k^{1/2}) > 0$.

These two problems really belong to a chapter called irregularities of distribution of diaphantine analysis, a subject to which K. F. Roth contributed many deep results.

8. Finally I state a few miscellaneous problems on polynomials. First an old problem of mine: Let $-1 \le x_1 < \ldots < x_n \le 1$. Let $\ell_k(x_k) = 1$, $\ell_k(x_i) = 0$ for $k \ne i$ be the fundamental functions of the Lagrange interpolation. Prove (or disprove) $(x_0 = -1,\ x_{n+1} = 1)$ that

$$(1) \qquad \min_{0 \le i \le n+1} \ \max_{x_i < x < x_{i+1}} \ \sum_{k=1}^{n} |\ell_k(x)| < c \log n .$$

I proved the much weaker result with $n^{1/2}$ instead of $c \log n$. Perhaps I overlooked a simple approach but I never got anywhere with (1).

The following questions can not be difficult and the answers are perhaps known: Let $z_1, \ldots, z_n$ be n points on the unit circle. Let $P_n(z) = \Pi_{i=1}^n (z - z_i)$, let y_i be the point on the arc (z_i, z_{i+1}) where $|P_n(z)|$ assumes its maximum. Is it true that

$$(2) \quad \prod_{i=1}^{n} |P_n(y_i)| \leq 2^n \text{ and } \sum_{i=1}^{n} |P_n(y_i)| \geq 2n?$$

There is equality in (2) for $z^n - 1$.

Let $|w_j| = 1$, $1 \leq j \leq n + 1$, $P_n(z) = \Pi_{i=1}^n (z - z_i)$, $|z_i| = 1$. But

$$A_n(w_1, \ldots, w_{n+1}) = \min_{P_n} \max_{1 \leq j \leq n+1} |P_n(w_j)|.$$

Is it true that $A_n(w_1, \ldots, w_{n+1})$ is maximal if $w_j^{n+1} = 1$? i.e., if the w_j's are the $(n+1)$-st roots of unity. Determine the extreme value. This surely must be simple but at the moment I do not know the answer.

Let $|z_i| \leq 1$, $1 \leq i \leq n$, $f_n(z) = \Pi_{i=1}^n (z - z_i)$. Put

$$A(f_n) = \max_{0 \leq r \leq 1} \max_{|z|=r} |f_n(z)| \min_{|z|=r} |f_n(z)|.$$

How large is $\max_{f_n} A(f_n)$?

Some of these questions may not be "serious" Mathematics but I am sure the following final problem considered by D. J. Newman and myself for a long time is both difficult and interesting: Let $\varepsilon_k = \pm 1$. Is it true that there is an absolute constant c so that for every choice of the ε_k's

$$\max_{|z|=1} \left| \sum_{k=1}^{n} \varepsilon_k z^k \right| > (1 + c) n^{1/2}?$$

This probably remains true if the condition $\varepsilon_k = \pm 1$ is replaced by $|\varepsilon_k| = 1$. For this section, see [1], [6].

References

1 Breusch, R., On the sum of relative extrema of $|f(z)|$ on the unit circle, Bull. Amer. Math. Soc. 53 (1947), 982-986.

2 Clunie, J., On a problem of Erdös, J. London Math. Soc. 42 (1967), 133-136.

3 Danzer, L. W. and Ch. Pommerenke, Über die Diskriminante von Mengen gegebene Durchmessers, Monatsh. Math. 71 (1967), 100-113.

4 Elbert, A., Über eine Vermutung von Erdös betreffs Polynome I, II. Studia Sci. Math. Hungar. 1 (1966), 119-128 and 3 (1968), 299-324.

5 Erdös, P. An extremum problem concerning tingonometric polynomials, Acta Sci. Math. (Szeged) 9 (1939), 113-115.

6 Erdös, P., Some remarks on polynomials, Bull. Amer. Math. Soc. 53 (1947), 1169-1176.

7 =(I) Erdös, P., F. Herzog and G. Piranian, Metric properties of polynomials, J. d'Analyse Math. 6 (1958), 125-148.

8 Erdös, P. and E. Netanyahu, A remark on polynomials and the transfinite diameter, Israel J. Math. 14 (1973), 23-25.

9 Goodman, A. W. and R. E. Goodman, A circle covering theorem, Amer. Math. Monthly 52 (1945), 494-498.

10 Kristiansen, G. K., Proof of an inequality for trigonometric polynomials, Proc. Amer. Math. Soc., 44 (1974), 49-57.

11 Kristiansen, G. K., Proof of a polynomial conjecture, Proc. Amer. Math. Soc. 44 (1974), 58-60.

12 =(PI) Pommerenke, Ch., On some problems by Erdös, Herzog and Piranian, Michigan Math. J. 6 (1959), 221-225.

13 Pommerenke, Ch., On some metric properties of polynomials with real zeros, I and II, Michigan Math. J., 6 (1959), 377-380 and 8 (1961), 49-54.

14 =(PII) Pommerenke, Ch., On metric properties of complex polynomials, Michigan Math. J. 8 (1961), 97-115.

15 Saff, E. B. and T. Scheill-Small, Coefficient and integral mean estimates for algebraic and trignometric polynomials with restricted zeros, J. London Math. Soc. 9 (1974), 16-22.

P. Erdös
Hungarian Academy of Sciences, Budapest, Hungary