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ON PRODUCTS OF FACTORIALS

BY

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Abstract. An old conjecture stated that (except for trivial cases) the product of consecutive integers is never an exact power. This conjecture was finally proved recently by Erdös and Selfridge. In the same spirit one can ask when the product of two or more disjoint blocks of consecutive integers can be a square or higher power. For example, if $A_1, \cdots, A_n$ are disjoint intervals each consisting of at least 3 integers then perhaps the product $\prod_{k=1}^{n} \prod_{a_k \in A_k} a_k$ is a nonzero square only in a finite number of cases.

In this paper we study products of factorials $\prod_k a_k!$. In particular, we investigate the equation

$$\prod_{k=1}^{t} a_k! = y^2, \tag{1}$$

especially in the case that n, the value of the largest a_k is given, and the minimum number of factorials is required. It turns out that each increase in the number t of factorials allowed rather dramatically increases the set of n for which (1) is solvable until the value $t=6$ is reached, after which *no* increase in the set of n occurs.

Introduction. An old conjecture stated that (except for trivial cases) the product of consecutive integers is never a power. This conjecture was finally proved recently by Erdös and Selfridge [3]. In the same spirit one can ask when the product of two or more disjoint blocks of consecutive integers can be a square or higher power. For example, if $A_1, \cdots, A_n$ are disjoint intervals each consisting of at least 3 integers then perhaps the product $\prod_{k=1}^{n} \prod_{a_k \in A_k} a_k$ is a square only in a finite number of cases.

In this paper we study products of factorials $\prod a_k!$. We prove that the number of distinct integers of the form $\prod_{a_1 < \cdots < a_t \le n} a_k!$ is

$$\exp\left\{(1+o(1)) \frac{n \log \log n}{\log n}\right\}.$$

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[1] The work on this paper was done while both authors were at the University of California, Los Angeles, California.

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especially in the case that n, the value of the largest a_k, is given and the minimum number of factorials is required. It turns out that each increase in the number t of factorials allowed rather dramatically increases the set of n for which (1) is solvable until the value $t = 6$ is reached after which *no* increase in the set of n occurs.

Finally, we mention several questions which we did not look at or were not able to resolve.

We make a few remarks concerning notation. All integers we consider will be positive. In general p_i will denote the ith prime, $p, q, q_1, q_2, \cdots$ will denote primes and δ, ε and c (possibly with subscripts) will denote suitably chosen positive constants. As usual $\pi(x)$ will denote the number of primes not exceeding x, $|X|$ will denote the cardinality of the set X and $[1, n]$ will denote the set $\{1, 2, \cdots, n\}$.

The number of products. For a subset $A \subseteq [1, n]$, let $m(A)$ denote the product

$$m(A) = \prod_{a \in A} a!$$

The following result shows that the set of possible values of $m(A)$ is rather sparse.

THEOREM 1.

$$m(n) = |\{m(A) : A \subseteq [1, n]\}| = \exp\left\{(1 + o(1)) \frac{n \log\log n}{\log n}\right\}. \tag{2}$$

Proof. (i) *Upper bound.* Write each product $\prod_{a \in A} a!$ in the form

$$\prod_{a \in A} a! = B \prod_k A_k = 2^{\alpha_1} 3^{\alpha_2} \cdots p_t^{\alpha_t}$$

where $t = \pi(n)$, A_k consists of all prime factors of the product belonging to $(n/2^{k+1}, n/2^k]$ for $0 \leq k \leq (2 \log\log n)/\log 2$ and all the remaining primes (i.e., those less than $n/\log^2 n$) divide B. Clearly

$$n^2 > \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_t .$$

Thus the number of choices for B is at most

$$(n^2)^{n/\log^2 n} = \exp(2n/\log n) .$$

The number of primes in the interval $(n/2^{k+1}, n/2^k]$ is $(1+o(1))(n/2^{k+1}\log n)$ (since $2^k \leq \log^2 n$). Since the α_j for these primes are all less than $2^{k+1}n$, the number of choices for A_k is at most

$$\binom{2^{k+1}n + \dfrac{(1+o(1))\,n}{2^{k+1}\log n}}{2^{k+1}n} < (c\cdot 2^{2k}\log n)^{(1+o(1))n/(2^{k+1}\log n)} .$$

Therefore the total number of choices for $\prod_k A_k$ is at most

$$(c\log n)^{(1+o(1))\,\Sigma_k n/(2^{k+1}\log n)} \prod_k (2^k)^{n/(2^k\log n)}$$
$$\cdot \exp\left\{(1+o(1))\,\frac{n\log\log n}{\log n}\right\}.$$

This estimate, combined with that for B, proves the upper bound in (2).

(ii) *Lower bound.* Define d_k by

$$d_k = p_k - p_{k-1} . \tag{3}$$

We first show

$$m(n) \geq \prod_{k=2}^{r} d_k , \tag{4}$$

where $r = \pi(n)$. For $A \subseteq [1, n]$, let

$$U(A) = (u_2(A), u_3(A), \cdots, u_r(A))$$

be defined by $u_k(A) = |\{a \in A : p_{k-1} \leq a < p_k\}|$. The definition of d_k implies that $u_k(A) \leq d_k$. On the other hand, for any sequence $w = (w_k, \cdots, w_r)$ with $w_k \leq d_k$, there exists a set $A_w \subseteq [1, n]$ with $u_k(A_w) = w_k$ for all k. Namely, we just choose

$$A_w = \bigcup_k \{v : p_{k-1} \leq v < p_{k-1} + w_k\} .$$

We claim

$$m(A_w) = m(A_{w'}) \Longrightarrow w = w' . \tag{5}$$

If $m = m(A_w) = m(A_{w'})$ then certainly $w_r = w'_r$, since this is just the power of p_{r-1} which occurs in m. Suppose that $m(A_w) = m(A_{w'})$ implies $w_i = w'_i$ for $i > k$. Then it is clear that the only way for the powers of p_{k-1} in $m(A_w)$ and $m(A_{w'})$ to be equal is to have $w_k = w'_k$. Thus, by induction, $w_k = w'_k$ for all k, and (5) follows. Since there are $\prod_{k=2}^{r} d_k$ choices for w, then (4) is proved.

Finally, to establish the lower bound in (2) it will be (more than) enough to show

$$\prod_{p_k \le n} d_k = \exp\,(1 + o(1))\,\frac{n \log\log n}{\log n}\,. \tag{6}$$

By the prime number theorem there are $(1 + o(1))(n/\log n)$ factors in the product in (6). Since $\sum_{p_k \le n} d_k = p_r$, the product is maximized when all the factors are equal. Thus

$$\begin{aligned}\prod_{p_k \le n} d_k \le \left(\frac{n}{\pi(n)}\right)^{\pi(n)} &= (\log n)^{(1+o(1))n/\log n} \\ &= \exp\,(1 + o(1))\,\frac{n \log\log n}{\log n}\,.\end{aligned} \tag{7}$$

To prove the inequality in the other direction we argue as follows. Write

$$\prod_{p_k \le n} d_k = \Pi_1\,\Pi_2\,,$$

where in Π_1 we take all the $d_k \le (\log n)/\log\log n$ and in Π_2 we take all the $d_k > (\log n)/\log\log n$. It is well known (and follows immediately from Brun's method) that the number of $d_k \le n$ satisfying $d_k = t$ is less than

$$\frac{cn}{\log^2 n}\prod_{p \mid t}(1 + 1/p) < \frac{c_1 n}{\log^2 n}\log\log t\,.$$

Thus the number of k for which $p_k \le n$ and $d_k \le (\log n)/\log\log n$ is less than

$$\frac{c_2 n \log\log\log n}{\log n\,\log\log n}\,.$$

Since Π_1 has $o(n/\log n)$ factors,

$$\begin{aligned}\prod_{p_k \le n} d_k \ge \Pi_2 &\ge \left(\frac{\log n}{\log\log n}\right)^{(1+o(1))n/\log n} \\ &\ge \exp\left\{(1 + o(1))\,\frac{n \log\log n}{\log n}\right\}.\end{aligned}$$

This proves (6) and the proof of (2) is complete. ∎

With a little more complicated argument, we could show that for all sufficiently large n,

$$\prod_{p_k \leq n} d_k < \exp\left(\frac{n \log\log n}{\log n}\right). \tag{8}$$

Perhaps it is true that (8) holds for all n. We can prove that $m(n)/\prod_{k=1}^{\pi(n)} d_k \to \infty$ but we do not give the proof, since we certainly cannot at present give an asymptotic formula for $m(n)$.

One could ask, for a fixed n, which choice of B with $|B| = n$ minimizes $|\{m(A) : A \subseteq B\}|$. Presumably it is $B = [1, n]$. Also, if $b(n)$ denotes

$$\max\{|B| : B \subseteq [1, n] \text{ and all } m(A) \text{ are distinct for all } A \subseteq B\},$$

then is it true that $b(n)/\pi(n) \to \infty$?

Products which are squares. For $k \geq 1$ define F_k by

$$F_k \equiv \{n : \text{for some } A \subseteq [1, n] \text{ with } \max_{a \in A}\{a\} = n \text{ and } |A| \leq k,\ m(A) = y^2 \text{ for some integer } y\}$$

and define D_k by

$$D_k \equiv F_k - F_{k-1},$$

where F_0 is defined to be empty. The main results of this paper deal with various properties of the F_k and D_k.

To begin with it is clear that for any prime p

$$p \notin D_k \text{ for any } k. \tag{9}$$

On the other hand, if n is composite, then n certainly belongs to $\bigcup_k D_k$. In fact:

(i) If $n = a^2$ then $n!(n-1)! = y^2$ and $n \in F_2$;

(ii) If $n = a^2 b$ with $a > 1$, $b > 1$ then $n!(n-1)!b!(b-1)! = y^2$ and $n \in F_4$;

(iii) If $n = ab$ and $a > 1$, $b > 1$, $a \neq b$, then $n!(n-1)!a!(a-1)!b!(b-1)! = y^2$ and $n \in F_6$. (If $|a - b| = 1$ then in fact $n \in F_4$.)

Thus we have

Fact 1. $D_k = \emptyset$ for $k > 6$.

Of course, it is immediate that $D_1 = \{1\}$. A result of Erdös-Selfridge [3] shows that no nontrivial product (i. e., having more than one term) of consecutive integers can be a square. This implies

Fact 2. $D_2 = \{n^2 : n > 1\}$.

Consequently, all integers excluding the primes and squares are partitioned into the four sets D_3, D_4, D_5 and D_6. We next examine each of these sets a little more carefully.

Three factors. It is easy to see that D_3 is somewhat larger than D_2. For observe that if $\bar{q}(x)$ denotes the *square-free* part of x, then for any $a > 1$ the integer $n = b^2\bar{q}(a!) \in D_3$ for b sufficiently large, since in this case

$$n!\,(n-1)!\,a! = y^2. \tag{10}$$

Another class of elements of D_3 is generated as follows. Write $a! = uv$ with $(u, v) = 1$. Let x and y be any solution of the Pell equation

$$ux^2 - vy^2 = 1,$$

and take $a_1 = ux^2$, $a_2 = vy^2 - 1 = a_1 - 2$. Thus

$$\begin{aligned} \bar{q}(a_1!\,a_2!\,a!) &= \bar{q}(a_1(a_1-1)\,uv) \\ &= \bar{q}(ux^2 \cdot vy^2\,uv) = 1 \end{aligned} \tag{11}$$

and so, when u is not a square, $a_1 \in D_3$. Perhaps there are just finitely many elements of D_3 which are not in either of these two classes. On the other hand, D_3 is still relatively sparse as the following result shows, where $S(n)$ denotes the number of elements of a set S which do not exceed n.

THEOREM 2. $D_3(n) = o(n)$.

Proof. Suppose $a_1 \in D_3$. Then there exist a_2 and a_3 with $a_1 > a_2 > a_3$ such that

$$a_1!\,a_2!\,a_3! = y^2.$$

Write $a_1 = a_2 + k$. Then we have

$$a_1(a_1-1)\cdots(a_1-k+1)\,a_3! = z^2. \tag{12}$$

The product of the primes in $(\frac{1}{2}a_3, a_3)$ exceeds $c_1 e^{(1/2+o(1))a_3}$. Since each of these primes occurs to the first power in $a_3!$, each must also divide $a_1(a_1-1)\cdots(a_1-k+1)$. Hence

$$a_1^k > c_1 e^{a_3/2}, \tag{13}$$

i. e.,

$$a_3 < c_2 k \log a_1 .$$

We shall use the following well-known result of Sylvester and Schur [1]:

Fact 3. The product of k consecutive integers $> k$ is divisible by some prime $p > k$.

Usually we can assume that p occurs only to the first power, as the next result shows.

Fact 4. The number of $n \leq x$ such that for some k the largest prime factor of $n(n-1)\cdots(n-k+1)$ occurs to a power >1 is $o(x)$.

Proof of Fact 4. Let us call an integer *bad* if it belongs to an interval $[a, a-1,\cdots, a-k+1]$ for some a and k such that the largest prime dividing $a(a-1)\cdots(a-k+1)$ occurs to a power ≥ 2. It suffices to prove that there are only $o(x)$ bad integers $n \leq x$. To do this, we use the following known result of Erdös [2].

Fact 5. A set of k consecutive integers always contains an integer which is either prime or divisible by a prime exceeding $ck \log k$.

Consider a fixed prime p and an interval I_k of length k containing a bad integer which is bad because of p. Thus p is the largest prime dividing an integer in I_k and some integer in I_k is divisible by p^α for some $\alpha \geq 2$. This implies that no integer in I_k can be prime. Thus, by Fact 5,

$$p > ck \log k ,$$

i. e.,

$$k < c_3 p/\log p . \tag{14}$$

Up to x there are at most x/p^2 multiples of p^2, so that the total number of bad integers $\leq x$ which are bad because of the prime p is at most

$$\frac{x}{p^2} \cdot \frac{c_3 p}{\log p} = \frac{c_3 x}{p \log p}. \tag{15}$$

However, this is being more generous than necessary with the small primes. For a large fixed integer D, it is not hard to obtain the estimate

$$|\{a \le x : \text{a has all prime factors } \le D\}| < c_4 \log^D x. \tag{16}$$

Any interval I'_k containing such an integer a and having all its terms with prime factors $\le D$ must have length $\le 2D$, for otherwise since there is a prime q with $D < q < 2D$ (by Chebyshev) and this q must divide some integer in I'_k, we would have a contradiction. The sum of the lengths of these I'_k is at most $c(D) \log^D x$, which is certainly $o(x)$.

Thus from (15) we have as an upper bound on the number of bad integers $\le x$ the sum

$$\sum_{D<p\le x} \frac{c_3 x}{p \log p} + o(x),$$

which is bounded above by $\varepsilon(D)\, x$, where $\varepsilon(D) \to 0$ as $D \to \infty$. This proves Fact 4. ∎

Continuing now the proof of Theorem 2, by Fact 2 we may assume the largest prime factor p of $a_1(a_1-1)\cdots(a_1-k+1)$ occurs to the first power. Furthermore, we may also assume that a_1 itself has a prime factor $> a_1^{\varepsilon}$ since those a_1 for which this does not occur have density $\le c(\varepsilon)$, where $c(\varepsilon)$ goes to zero with ε. Thus

$$p > a_1^{\varepsilon}. \tag{17}$$

Also, since $p > k$ by Fact 3, p divides exactly *one* of the k integers $a_1, a_1 - 1, \cdots, a_1 - k + 1$. Since p only occurs to the first power, (12) implies that p must divide $a_3!$ and so

$$a_3 \ge p. \tag{18}$$

Therefore by (13), (17) and (18)

$$k > c_4\, a_3/\log a_1 > c_4\, a_1^{\varepsilon}/\log a_1 > c_5\, a_1^{\varepsilon/2}. \tag{19}$$

Also, by a well-known result of Huxley [4], we must have $a_1 > k^{3/2}$ (since otherwise $[a_1, a_1 - k]$ will contain a prime). We may now apply the following result of Ramachandra [5]:

THEOREM. *Let* $k^{3/2} \leq u \leq k^{\log\log k}$ *Then the largest prime divisor* $P(u, k)$ *of* $\prod_{i=1}^{k}(u+i)$ *satisfies*

$$P(u, k) > k^{1+2^{\lambda(u,k)}},$$

where $\lambda(u, k) = -((\log u)/(\log k) + 8)$.

In particular, for some $\delta = \delta(\varepsilon) > 0$, there is a prime $q > k^{1+\delta}$ dividing $a_1(a_1 - 1)\cdots(a_1 - k + 1)$. Consequently, by (18),

$$a_3 \geq p \geq q > k^{1+\delta}. \tag{20}$$

Also by (13) and (19),

$$a_3 < c_2\, k \,\log a_1 < c_1(\varepsilon)\, k\, \log k. \tag{21}$$

However, (19), (20) and (21) are clearly inconsistent for large x (and we may assume, for example, that $a_1 > x^{1/2}$). Thus, except for $o(x)$ integers $a_1 \leq x$, (12) is impossible. This proves Theorem 2. ∎

Suppose we call an integer n *bad'* if its greatest prime factor $P(n)$ occurs with an exponent >1. An old result of one of the authors states that the number of bad' $n < x$ is

$$xe^{-(1+o(1))(\log x \log\log x)^{1/2}}.$$

No doubt almost all bad' numbers n are bad' because they have $P(n)^2 \| n$.

One can modify bad'ness as follows: Call n *bad''* if it occurs in some interval $[a-k, a]$ such that all prime factors $>k$ of $\prod_{i=0}^{k}(a-i)$ occur with an exponent >1. It seems likely that the number $B(x)$ of bad'' integers $\leq x$ not only satisfies $B(x)/\sqrt{x} \to c$, but in fact is asymptotic to the number of "powerful" numbers $\leq x$, (i.e., numbers with all prime factors occurring to a power >1).

The two classes of examples of elements of D_3 given at the beginning of this section both have

$$a_2 \geq a_1 - 2.$$

Examples do exist for which $a_2 = a_1 - 3$, e.g., 10! 7! 6!, 50! 47! 3! and 50! 47! 4! are all squares. Are there others? Can $a_1!\, a_2!\, a_3!$

ever be a square for $a_3 < a_2 < a_1 - 3$? It is not difficult to show that if $a_1 \in D_3$ and $a_1 = 2p$ for some odd prime p, then a_1 is either 6 (with $6!\,5!\,3! = y^2$) or a_1 is 10 (with $10!\,7!\,6! = y^2$). We are sure now that $D_3(x) = (c + o(1))\,x^{1/2}$ but we cannot prove it.

Four factors. To begin with, observe that if n has a nontrivial square factor, say $n = m^2 r$ with $m > 1$, then

$$\bar{q}(n!(n-1)!\,r!(r-1)!) = \bar{q}(nr) = 1$$

so that $n \in F_4$. Thus all multiples of 4 belong to F_4 and so D_4 has positive density and we have by Theorem 2

Fact 6. $\lim_{n\to\infty} D_4(n)/D_3(n) = \infty$.

On the other hand, there are certainly squarefree elements of F_4. For example, if a and $a+1$ are both squarefree then by setting

$$a_1 = a(a+1)\,,$$
$$a_2 = a(a+1) - 1\,,$$
$$a_3 = a + 1\,,$$
$$a_4 = a - 1\,,$$

we have $\bar{q}(a_1!\,a_2!\,a_3!\,a_4!) = \bar{q}(a(a+1)\cdot(a+1)\,a) = 1$ and so $a_1 \in F_4$. However, squarefree integers of this form are relatively rare and in fact it seems likely that almost all squarefree integers do not belong to F_4. It can be shown that for any fixed prime q, almost all n of the form pq, p prime, do not belong to F_4. The proof uses the previously mentioned result of Ramachandra and we do not give it here.

Five factors. It was first pointed out by E. G. Straus (oral communication) that those n with a very small prime factor all belong to F_5. The precise statement of this is given as follows.

Fact 7. If $p \in \{2, 3, 5, 7, 11\}$ is a proper divisor of n, then $n \in F_5$.

Proof. We simply observe that each of the five expressions is a square:

$$(2m)!\,(2m-1)!\,(m)!\,(m-1)!\,2!\,,$$
$$(3m)!\,(3m-1)!\,(2m)!\,(2m-1)!\,3!\,,$$
$$(5m)!\,(5m-1)!\,(m)!\,(m-1)!\,6!\,,$$
$$(7m)!\,(7m-1)!\,(5m)!\,(5m-1)!\,7!\,,$$
$$(11m)!\,(11m-1)!\,(7m)!\,(7m-1)!\,11!\,. \quad \blacksquare$$

On the other hand, the prime 13 (as well as all larger primes) behaves differently as the following result indicates.

THEOREM 3. *For almost all primes p,*

$$13p \notin F_5\,. \tag{22}$$

Proof. Suppose $13q \in F_5$ for a large prime q. By the comment at the end of the preceding section that $13p \notin F_4$ for almost all primes p, we may assume $13q \in D_5$. Thus there exist $a_1 = 13q > a_2 > a_3 > a_4 > a_5$ such that

$$a_1!\,a_2!\,a_3!\,a_4!\,a_5! = y^2\,. \tag{23}$$

From (23) we have

$$\underbrace{a_1(a_1-1)\cdots(a_2+1)}_{I_1}\ \underbrace{a_3(a_3-1)\cdots(a_4+1)}_{I_2}\ a_5! = z^2\,, \tag{24}$$

where I_1 and I_2 are defined to be the intervals $\{a_1,\cdots,a_2+1\}$ and $\{a_3,\cdots,a_4+1\}$, respectively.

Fact 8.

$$a_5 < c a_1^{2/3}\,. \tag{25}$$

Proof of Fact 8. In (24) no prime can occur in I_1. Thus, by a result of Huxley [4],

$$a_1 - a_2 < a_1^{3/5}\,. \tag{26}$$

Of course, we may assume $a_4 > a_1^{2/3}$, since otherwise we are done. Now for large a_1 there are two possibilities:

(i) If $a_3 - a_4 \geq a_4^{3/5}$ then by Huxley [4] we have

$$a_1^{a_1^{3/5}} \geq \prod_{\substack{a_4 < p \leq a_3 \\ p \text{ prime}}} p > c_1\, e^{(1+o(1))(a_4-a_3)}\,,$$

since any prime $p \in I_2$ must divide at least one integer in I_1 in order for (24) to hold. Therefore

$$a_3 - a_4 < c_2\, a_1^{3/5} \log a_1 .$$

(ii) If $a_3 - a_4 < a_4^{3/5}$ then automatically we have $a_3 - a_4 < a_1^{3/5}$. Hence in either case

$$\prod_{a_4 < x < a_3} x < a_3^{c_2 a_1^{3/5}} \log a_1 .$$

But

$$\prod_{\substack{a_5/2 < p \leq a_5 \\ p \text{ prime}}} p > c_3\, e^{(1/2 + o(1))\, a_5}$$

and any prime $p \in (a_5/2,\, a_5]$ must divide some integer in $I_1 \cup I_2$. Therefore

$$c_3\, e^{a_5/2} < a_3^{c_2 a_1^{3/5} \log a_1} \cdot a_1^{a_1^{3/5}} ,$$

$$a_5 < c_4\, a_1^{3/5} \log^2 a_1 < c\, a_1^{2/3}$$

for a suitable constant c. ∎

Note that the same argument applies to the product

$$a_1!\, a_2! \cdots a_{2r+1}! \text{ for any fixed } r,$$

where the constant c now depends on r.

Fact 9. For almost all(2) primes p, all of the expressions $13p \pm 1,\ 12p \pm 1,\ 11p \pm 1, \cdots,\ p \pm 1$ have a prime factor exceeding p^{ε}.

Proof. We only give the argument for $p - 1$ (the other cases are similar). Denote by $\pi_{\varepsilon}(x)$ the number of primes $p \leq x$ for which all prime factors of $p - 1$ are $\leq x^{\varepsilon}$. Put

$$A_{\varepsilon}(n) = \prod_{\substack{p^{\alpha} \| n \\ p \leq x^{\varepsilon}}} p^{\alpha} ,$$

where $p^{\alpha} \| n$ denotes the fact that p^{α} divides n but $p^{\alpha+1}$ does not divide n. Then

$$\prod_{p \leq x} A_{\varepsilon}(p - 1) \leq \prod_{q \leq x^{\varepsilon}} q^{(\alpha_q + \alpha_{q^2} + \cdots)} ,$$

(2) I.e., for all but $c_{\varepsilon}\, \pi(x)$ primes $\leq x$, where $c_{\varepsilon} \to 0$ as $\varepsilon \to 0$.

where α_{q^r} is the number of primes $p \leq x$ with $p \equiv 1 \pmod{q^r}$. By Brun-Titchmarsh [6] we have for $q^r \leq x^{1/2}$,

$$\alpha_{q^r} < \frac{cx}{q^r \log x},$$

and for $q^r < x$,

$$\alpha_{q^r} \leq \frac{x}{q^r}.$$

Thus

$$\prod_{p \leq x} A_\varepsilon(p-1) < \Big(\prod_{q \leq x^\varepsilon} q^{1/q}\Big)^{cx/\log x} \prod_{\substack{q \leq x^\varepsilon \\ q^r \geq x^{1/2}}} q^{x/q^r}. \tag{27}$$

But

$$\prod_{q \leq x^\varepsilon} q^{1/q} = \exp\Big(\sum_{q \leq x^\varepsilon} \frac{\log q}{q}\Big) < x^{c_1 \varepsilon}$$

and

$$\prod_{\substack{q \leq x^\varepsilon \\ q^r \geq x^{1/2}}} q^{x/q^r} < x^{x \Sigma' 1/q^r} < x^{x^{1/2+\varepsilon}}$$

where in Σ', $q \leq x^\varepsilon$, $q^r \geq x^{1/2}$ and so

$$\sum{}' 1/q^r \leq x^{\varepsilon - 1/2}.$$

Therefore

$$\begin{aligned}\prod_{p \leq x} A_\varepsilon(p-1) &< (x^{c_1\varepsilon})^{cx/\log x} x^{x^{1/2+\varepsilon}}, \\ &< e^{c_2 \varepsilon x},\end{aligned}$$

which easily implies

$$\pi_\varepsilon(x) < c_3 \varepsilon x/\log x. \tag{28}$$

Similar arguments give the inequalities corresponding to (28) for $13p \pm 1$, $12p \pm 1$, etc., and by the prime number theorem, Fact 9 follows. ▌

It follows from (24) that I_1 cannot contain a prime. Hence the only multiple of q which can belong to I_1 is $a_1 = 13q$. But Fact 8 implies that $q > a_5$. Thus, by (24), I_2 contains at least one multiple of q, say aq.

Fact 10. I_2 contains exactly one multiple of q.

Proof of Fact 10. Suppose I_2 contains at least two multiples of q. Then $|I_2| > q$, so that

$$\prod_{p \in I_2} p > ce^{q/2}.$$

Since any prime $p \in I_2$ must divide some element of I_1, then

$$(13q)^{|I_1|} > ce^{q/2},$$

i. e.,

$$|I_1| > \frac{c_1 q}{\log q}.$$

By the previously mentioned result of Huxley [4], this implies that I_1 contains a prime, which is impossible. Since we have seen that I_2 contains some multiple aq of q, Fact 10 is proved. ∎

Fact 11. $|I_1| + |I_2| \geq 3$.

Proof of Fact 11. Suppose $|I_1| = |I_2| = 1$. Then

$$I_1 = \{13q\} \text{ and } I_2 = \{aq\} \text{ for some } a < 13.$$

By (24),

$$13aa_5! = x^2 \tag{29}$$

must hold for some a_5 and x. This forces $a_5 \leq 16$. A check of all these cases, however, reveals that (29) is in fact impossible. Since $|I_1|$ and $|I_2|$ are positive, Fact 11 is proved. ∎

Fact 12. For any $a \leq 13$ and any m and almost all primes p, the largest prime factor p' of $\prod_{k=-m}^{m} (ap - 13k - 1)$ occurs to the first power.

Proof of Fact 12. By Fact 9 and the prime number theorem, it will be enough to prove that there are fewer than $x^{1-\delta}$ primes $p \leq x$ for which the largest prime divisor p' of $\prod_{k=-m}^{m}(ap - 13k - 1)$ satisfies:

(i) $p' > x^{\varepsilon}$,

(ii) $(p')^2 | \prod_{k=-m}^{m} (ap - 13k - 1)$.

Consider the arithmetic progression $I = \{ap - 13k - 1 : -m \leq k \leq m\}$ and let us estimate for a fixed $p' > x^{\varepsilon}$ the number of primes $p \leq x$ satisfying (ii). For such a p, $(p')^2$ divides some element of I, say $u(p')^2$, and we let d denote $|ap - u(p')^2|$. By the previously mentioned result of Ramachandra, modified to apply to arithmetic progressions (he informs us that his proof gives this without essential change), we obtain

$$d < (p')^{1-\eta} \tag{30}$$

for some $\eta = \eta(\varepsilon) > 0$. Thus for a fixed p', since there are just $x/(p')^2$ multiples of $(p')^2$ less than x, there are at most

$$2d \cdot \frac{x}{(p')^2} < \frac{cx}{(p')^{1+\eta}}$$

possible values of p satisfying (ii). Hence the *total* number of these p for all p' satisfying (i) is at most

$$\sum_{p'>x^\varepsilon} \frac{cx}{(p')^{1+\eta}} \le x \sum_{m>x^\varepsilon} \frac{c}{m^{1+\eta}} < \frac{cx}{x^{\varepsilon\eta}} < x^{1-\delta}.$$

This proves Fact 12. ∎

Of course, the same conclusion holds for $\prod_{k=-m_1}^{m_2}(ap - 13k - 1)$ as well as for $\prod_{k=-m_1;\ k\neq 0}^{m_2}(ap - k)$, where in the second product $m_1 + m_2 \ge 1$. Thus we may henceforth assume that the largest prime p' dividing any element $\equiv -1 \pmod{13}$ in $I_1 \cup I_2$ occurs to the first power and the largest prime p'' dividing any element in $I_2 - \{aq\}$ occurs to the first power. Define p^* by

$$p^* = \begin{cases} p'' & \text{if } |I_1| = 1, \\ p' & \text{if } |I_1| > 1. \end{cases}$$

It follows from Fact 9 that we may assume

$$p^* > q^\varepsilon.$$

Fact 13. p^* divides at most one element of $I_1 \cup I_2$.

Proof of Fact 13. Suppose p^* divides at least two elements of $I_1 \cup I_2$. There are two possibilities.

(a) Suppose $|I_1| = 1$. Then $|I_2| > p^* > q^\varepsilon$ so that by Ramachandra [5] there must be a prime divisor of $\prod_{x\in I_2;\ x\neq aq} x$ exceeding

$$|I_2|^{1+\delta} > (p^*)^{1+\delta},$$

which is a ***contradiction*** to the definition of p^*.

(b) Suppose $|I_1| > 1$. Then $p^* = p'$, and if p^* divides two elements of I_k for either $k = 1$ or $k = 2$, we reach a contradiction by the arguments used in case (a). Hence we must have p^* dividing some element of I_1 and some element of I_2, say

$$u_1 p^* = 13(q - d_1) - 1 \in I_1 ,$$

$$u_2 p^* = aq \pm d_2 \in I_2 ,$$

where $d_2 \neq 0$. Thus

$$p^*(13u_2 - au_1) = 13ad_1 \pm 13d_2 - a . \tag{31}$$

If $13u_2 - au_1 = 0$ then $13 | a$, which is a contradiction. Thus we may assume

$$13u_2 - au_1 \neq 0 . \tag{32}$$

By (31) this implies

$$d_1 + d_2 > cp^* , \quad \text{i.e.,} \quad |I_1| + |I_2| > c_1 p^* . \tag{33}$$

Again by Ramachandra we conclude that there must be a prime divisor of the integers $\equiv -1 \pmod{13}$ in some I_k exceeding

$$|I_k|^{1+\delta} > (c_2 p^*)^{1+\delta} ,$$

which is impossible. This proves Fact 13. ▌

From Fact 13 and the assumption that p^* occurs only to the first power in its multiple in $I_1 \cup I_2$, we must have $a_5 \geq p^*$. As before, since all primes in $(a_5/2, a_5)$ must divide elements in $I_1 \cup I_2$, then

$$(13q)^{|I_1|+|I_2|} \geq \prod_{x \in I_1 \cup I_2} x \geq \prod_{a_5/2 < p < a_5} p > e^{(1/2+o(1))a_5} \geq ce^{p^*/2} , \tag{34}$$

i.e.,

$$|I_1| + |I_2| > \frac{c_1 p^*}{\log q} > \frac{c_1 \varepsilon p^*}{\log p^*} .$$

Finally, once more by Ramachandra, we conclude that either $I_2 - \{aq\}$ has an element with a prime divisor exceeding $(c_2 \varepsilon p^*/(\log p^*))^{1+\eta}$ or $I_1 \cup I_2$ has an element $\equiv -1 \pmod{13}$ with a prime divisor exceeding $(c_2 \varepsilon p^*/(\log p^*))^{1+\eta}$. However, this is impossible for large q, since it contradicts the definition of p^*.

Thus the assumption that $13q \in F_5$ has led to a contradiction for almost all primes q. This completes the proof of Theorem 3. ▌

The reason this proof works for 13 (and all larger primes) but not for 2, 3, 5, 7 or 11 is because Fact 11 fails to hold for these smaller primes, i.e., (24) has solutions when 13 is replaced by a smaller prime.

It seems certain that almost all products of two primes do not belong to F_5.

The following problem may be of interest here. Consider the following two-variable sieve: Omit all integers n which are congruent to $i(\bmod p^2)$ for some i with $|i| < cp$ where $p > p_k$. For which k and c are there infinitely many integers which are not omitted?

Also we might ask whether it is true that for every k there exist k consecutive integers each having its greatest prime factor occurring to a power greater than 1.

Six factors. As pointed out earlier, every composite n belongs to F_6. It is of interest to determine the least element n^* of $D_6 = F_6 - F_5$.

Fact 14. $n^* = 527 = 17 \cdot 31$.

Proof. By Fact 7, no element of D_6 can be divisible by 2, 3, 5, 7 or 11. The remaining composite numbers less than 527 are listed below

TABLE 1

n	Square product of factorials
$221 = 13 \cdot 17$	221! 220! 18! 11! 7!
$247 = 13 \cdot 19$	247! 246! 187! 186! 20!
$299 = 13 \cdot 23$	299! 298! 27! 22!
$323 = 17 \cdot 19$	323! 322! 20! 14! 6!
$377 = 13 \cdot 29$	377! 376! 29! 23! 10!
$391 = 17 \cdot 23$	391! 389! 24! 21! 17!
$403 = 13 \cdot 31$	403! 402! 33! 30! 14!
$437 = 19 \cdot 23$	437! 436! 51! 49! 28!
$481 = 13 \cdot 37$	481! 479! 38! 33! 22!
$493 = 17 \cdot 29$	493! 491! 205! 202! 7!

The fact that $527 = 17 \cdot 31$ has no such representation can be verified by a direct (but lengthy) computation. ∎

We are reasonably certain that

$$D_6(n) > cn .$$

Miscellaneous remarks. Let A denote the set $\{a : a$ is squarefree and $abk! = y^2$ for some y, b, k with $a > b > k\}$. Of course, $A \subseteq F_5$ since for any $a \in A$, $a!(a-1)!\,b!(b-1)!\,k!$ is a square. Note that $a \in A$ implies $ta \in A$ since $(ta)(tb)\,k!$ is a square if $abk!$ is a square. It can be shown that

$$\sum_{a \in A} \frac{1}{a} < \infty ,$$

so that the density of the nonmultiples of the a's exists and is positive.

If eight factors are allowed, then for almost all n we can find *nearly equal* factorials, the largest being $n!$, whose product is a square. Specifically, for $n = n_1 n_2$, set

$$\begin{aligned} a_1 &= n = n_1 n_2 , & a_2 &= a_1 - 1 , \\ a_3 &= (n_1 - 1)\, n_2 , & a_4 &= a_3 - 1 , \\ a_5 &= n_1(n_2 - 1) , & a_6 &= a_5 - 1 , \\ a_7 &= (n_1 - 1)(n_2 - 1) , & a_8 &= a_7 - 1 . \end{aligned}$$

Then

$$\prod_{k=1}^{8} a_k! = y^2 ,$$

and a_8/a_1 is essentially equal to $(1 - 1/n_1)(1 - 1/n_2)$. Since for almost all n we can take $n_1 > n^\varepsilon$, $n_2 > n^\varepsilon$, the assertion follows.

Finally, one could ask the preceding questions for cubes and higher powers instead of squares. For example, it is not hard to show that for any k there is an $m(k)$ such that $\prod_{i=1}^{m} a_i! = y^k$ has infinitely many solutions for some $m \leq m(k)$. These, however, we leave for a later paper.

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