1 §. INTRODUCTION

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with $P(X_1 = 0) = P(X_1 = 1) = \frac{1}{2}$ and let $S_0 = 0$, $S_n = X_1 + X_2 + \ldots + X_n \ (n = 1, 2, \ldots)$ and

$$I(N, K) = \max_{0 \leq n \leq N - K} (S_{n+K} - S_n) \quad (N \geq K).$$

Define the r.v.'s $Z_N \ (N = 1, 2, \ldots)$ as follows: let $Z_N$ be the largest integer for which

$$I(N, Z_N) = Z_N.$$

This $Z_N$ is the length of the longest head-run. Studying the properties $Z_N$ resp. $I(N, K)$ Erdős and Rényi proved the following:

**Theorem A.** ([1]) Let $0 < C_1 < 1 < C_2 < \infty$ then for almost all $\omega \in \Omega \ (\Omega$ is the basic space) there exists a finite $N_0 = N_0(\omega, C_1, C_2)$
such that

\[ [C_1 \log N] \leq Z_N \leq [C_2 \log N] \]

if \( N \geq N_0 \).

The aim of this paper is to get sharper bounds of \( Z_N \). In connection with this problem our first result is

**Theorem 1.** Let \( \varepsilon \) be any positive number. Then for almost all \( \omega \in \Omega \) there exists a finite \( N_0 = N_0(\omega, \varepsilon) \) such that

\[ Z_N \geq [\log N - \log \log \log N + \log \log \varepsilon - 2 - \varepsilon] = \alpha_1(N) = \alpha_1 \]

if \( N \geq N_0 \).

This result is quite near to the best possible one in the following sense:

**Theorem 2.** Let \( \varepsilon \) be any positive number. Then for almost all \( \omega \in \Omega \) there exists an infinite sequence \( N_i = N_i(\omega, \varepsilon) \) \((i = 1, 2, \ldots)\) of integers such that

\[ Z_{N_i} < [\log N_i - \log \log \log N_i + \log \log \varepsilon - 1 + \varepsilon] = \alpha_2(N) = \alpha_2. \]

Theorems 1 and 2 together say that the length of the longest head-run is larger than \( \alpha_1 \) but in general not larger than \( \alpha_2 \). Clearly enough for some \( N \) the length of the longest head-run can be much larger than \( \alpha_2 \). In our next theorems the largest possible values of \( Z_N \) are investigated.

**Theorem 3.** Let \( \{\gamma_n\} \) be a sequence of positive numbers for which

\[ \sum_{n=1}^{\infty} 2^{-\gamma_n} = \infty. \]

Then for almost all \( \omega \in \Omega \) there exists an infinite sequence

\( N_i = N_i(\omega, \{\gamma_n\}) \) \((i = 1, 2, \ldots)\) of integers such that

\[ Z_{N_i} > \gamma_{N_i}. \]

This result is the best possible in the following sense:

\*Here and in what follows \( \log \) means logarithm with base \( 2; [x] \) is the integral part of \( x \).
Theorem 4. Let \( \{\delta_n\} \) be a sequence of positive numbers for which
\[
\sum_{n=1}^{\infty} 2^{-\delta_n} < \infty.
\]
Then for almost all \( \omega \in \Omega \) there exists a positive integer
\( N_0 = N_0(\omega, \{\delta_n\}) \) such that
\[
Z_N < \delta_N
\]
if \( N \geq N_0 \).

Theorems 1-4 are characterizing the length of the longest run containing no tail at all. One can ask about the length of the longest run containing at most \( T \) tails. In order to formulate our results precisely introduce the following notation: Let \( Z_N(T) \) be the largest integer for which
\[
l(N, Z_N(T)) \geq Z_N(T) - T.
\]
This \( Z_N(T) \) is the length of the longest run containing at most \( T \) tails.

Our Theorems 1-4 can be easily generalized for this case as follows:

**Theorem 1*.** Let \( \varepsilon \) be any positive number. Then for almost all \( \omega \in \Omega \) there exists a finite \( N_0 = N_0(\omega, T, \varepsilon) \) such that
\[
Z_N(T) \geq [\log N + T \log \log N - \log \log \log N - \log T! + \log \log \varepsilon - 2 - \varepsilon] = \alpha_1(N, T)
\]
if \( N \geq N_0 \).

**Theorem 2*.** Let \( \varepsilon \) be any positive number. Then for almost all \( \omega \in \Omega \) there exists an infinite sequence \( N_i = N_i(\omega, T, \varepsilon) \) of integers such that
\[
Z_{N_i}(T) < \alpha_2(N_i, T) = [\log N_i + T \log \log N_i - \log \log \log N_i - \log T! + \log \log \varepsilon - 1 + \varepsilon].
\]

**Theorem 3*.** Let \( \{\gamma_n\} \) be a sequence of positive integers for which
\[ \sum_{n=1}^{\infty} \gamma_n T^{2^{-\gamma_n}} = \infty. \]

Then for almost all \( \omega \in \Omega \) there exists an infinite sequence
\[ N_i = N_i(\omega, T, \{\gamma_n\}) \]

of integers such that
\[ Z_{N_i}(T) \geq \gamma_{N_i}. \]

**Theorem 4*. Let \( \{\delta_n\} \) be a sequence of positive integers for which
\[ \sum_{n=1}^{\infty} \delta_n T^{2^{-\delta_n}} < \infty. \]

Then for almost all \( \omega \in \Omega \) there exists a positive integer
\[ N_0 = N_0(\omega, T, \{\delta_n\}) \]

such that
\[ Z_N(T) < \delta_N \]

if \( N \geq N_0. \)

The last two Theorems clearly can be reformulated as follows:

**Theorem 3**. Let \( \{\gamma_n\} \) be a sequence of positive integers for which
\[ \sum_{n=1}^{\infty} \gamma_n T^{2^{-\gamma_n}} = \infty. \]

Then for almost all \( \omega \in \Omega \) there exists a sequence \( N_i = N_i(\omega, \{\gamma_n\}) \)

of integers such that
\[ S_{N_i} - S_{N_i - \gamma_{N_i}} \geq \gamma_{N_i} - T. \]

**Theorem 4**. Let \( \{\delta_n\} \) be a sequence of positive integers for which
\[ \sum_{n=1}^{\infty} \delta_n T^{2^{-\delta_n}} < \infty. \]

Then for almost all \( \omega \in \Omega \) there exists a positive integer
\[ N_0 = N_0(\omega, T, \{\delta_n\}) \]

such that
\[ S_N - S_{N - \delta_N} < \delta_N - T \]

if \( N \geq N_0. \)
2§. A THEOREM ON THE DISTRIBUTION OF $I(N, K)$

The proofs of Theorems 1-4 are based on the following

**Theorem 5.** We have

\[
\left(1 - 2^{-K-1} \frac{K^T + 1}{T!} (1 + o_K(1))\right) \left[\frac{N - 2K}{K}\right] + 1 \leq \leq \ \leq \ \leq \ P(I(N, K) < K - T) \leq \leq \leq \left(1 - 2^{-K-1} \frac{K^T + 1}{T!} (1 + o_K(1))\right) \left[\frac{1}{2} \left[\frac{N - 2K}{K}\right]\right] + 1
\]

if $N \geq 2K$.

Before the proof of this Theorem we prove our

**Lemma 1.** We have

\[
P(I(2N, N) \geq N - T) = \begin{cases} 
2^{-N-1}(N + 2) & \text{if } T = 0, \\
2^{-N-1}(N^2 + 4 - 2^{-N+1}) & \text{if } T = 1, \\
2^{-N-1} \frac{N^T + 1}{T!} (1 + o(1)) & \text{if } T > 1.
\end{cases}
\]

**Proof.** Let

\[A = A(T) = \{I(2N, N) \geq N - T\},\]

\[A_k = A_k(T) = \{S_{k+N} - S_k \geq N - T\} \quad (k = 0, 1, 2, \ldots, N),\]

and

\[S_{-j} = -\infty \quad (j = 1, 2, \ldots).\]

Then we clearly have

\[A = A_0 + A_0A_1 + A_0A_1A_2 + \ldots + A_0A_1\ldots A_{N-1}A_N\]

where
\[ P(A_0) = \sum_{j=0}^{T} \binom{N}{j} 2^{-N}, \]

and

\[ p_k = P(\bar{A}_0 \bar{A}_1 \ldots \bar{A}_{k-1} A_k) = \]

\[ = \sum_{k+1 \leq l_1 < l_2 < \ldots < l_T < k+N} P(\bar{A}_0 \bar{A}_1 \ldots \bar{A}_{k-1} A_k), \]

\[ X_k = X_{l_1} = X_{l_2} = \ldots = X_{l_T} = 0 \]

\[ = \sum_{k+1 \leq l_1 < l_2 < \ldots < l_T < k+N} P(A_k, X_k = X_{l_1} = X_{l_2} = \ldots \]

\[ \ldots = X_{l_T} = 0, S_{k-1} - S_{l_T-N-1} < k - l_T + N, \]

\[ S_{k-1} - S_{l_{T-1}-N-1} < k - l_{T-1} + N - 1, \ldots \]

\[ \ldots, S_{k-1} - S_{l_1-N-1} < k - l_1 + N - (T-1)) = \]

\[ = 2^{-N-1} \sum_{k+1 \leq l_1 < l_2 < \ldots < l_T < k+N} P(S_{k-1} - S_{l_{T-1}-N-1} < k - l_T + N, S_{k-1} - S_{l_{T-1}-N-1} < k - l_{T-1} + N - 1, \ldots \]

\[ \ldots, S_{k-1} - S_{l_1-N-1} < k - l_1 + N - (T-1)). \]

Especially if

(i) \( T = 0 \) then \( p_k = 2^{-N-1} \)

(ii) \( T = 1 \) then \( p_k = 2^{-N-1}(N - 2 + 2^{-k+1}) \)

(iii) \( T > 1 \) then \( p_k = 2^{-N-1} \left( \frac{N}{T} \right) (1 + o(1)) \)

what clearly implies our Lemma.

**Proof of Theorem 5.** Let

\[ B_k = \{ S_{k+K} - S_k \geq k - T \} \quad (k = 0, 1, 2, \ldots, N - K), \]

\[ C_l = \sum_{k=\lceil lK \rceil}^{(l+1)K} B_k \quad \left( l = 0, 1, 2, \ldots, \left\lceil \frac{N-2K}{K} \right\rceil \right), \]
Then by Lemma 1

\[ P(C_i) = 2^{-K-1} \frac{K^{T+1}}{T!} (1 + o_K(1)) \]

and since the events \( C_0, C_2, \ldots \) are independent we have

\[
P(\bar{D}_0) = P(\bar{C}_0)P(\bar{C}_2)\ldots = P(\bar{C}) 2^{\left[ \frac{1}{2} \left( \frac{N-2K}{K} \right) \right] + 1} = \left( 1 - 2^{-K-1} \frac{K^{T+1}}{T!} (1 + o_K(1)) \right) \left[ \frac{1}{2} \left( \frac{N-2K}{K} \right) \right] + 1
\]

and similarly

\[
P(\bar{D}_1) = \left( 1 - 2^{-K-1} \frac{K^{T+1}}{T!} (1 + o_K(1)) \right) \left[ \frac{1}{2} \left( \frac{N-2K}{K} \right) \right] + 1
\]

Clearly

\[ D_0 \subset \{ I(N,K) \geq K-T \} = D_0 + D_1 \]

and

\[ P(I(N,K) < K-T) = P(\bar{D}_0 + \bar{D}_1) = P(\bar{D}_0 \bar{D}_1) \geq P(\bar{D}_0)P(\bar{D}_1) \]

which proves Theorem 5. The right side of the last inequality follows from

the simple inequality

\[ P(D_1 | B_k) \geq P(D_1) \quad (k = 0, 1, 2, \ldots, N-K). \]

§3. THE PROOFS OF THEOREMS 1* - 4*

The following two Lemmas are trivial consequences of Theorem 5.

Lemma 2. Let \( N_j = N_j(T) \) be the smallest integer for which \( \alpha_1(N_j, T) = j \). Then
\[
\sum_{j=1}^{\infty} P(Z_{N_j}(T) < \alpha_1(N_j, T)) = \\
= \sum_{j=1}^{\infty} P(I(N_j, \alpha_1(N_j, T)) < \alpha_1(N_j, T) - T) < \infty.
\]

**Lemma 3.** Let \( \delta \) be a positive number and let \( N_j = N_j(T, \delta) \) be the smallest integer for which \( \alpha_2(N_j, T) = \lfloor j^{1+\delta} \rfloor \). Then
\[
\sum_{j=1}^{\infty} P(I(N_j, \alpha_2(N_j, T)) < \alpha_2(N_j, T) - T) = \infty
\]
if \( \delta \) is small enough.

Now Theorem 1* follows immediately from Lemma 2.

In order to prove Theorem 2* the following version of the Borel--Cantelli lemma will be applied:

**Lemma A.** ([2]) *If \( A_1, A_2, \ldots \) are arbitrary events, fulfilling the conditions
\[
\sum_{n=1}^{\infty} P(A_n) = \infty
\]
and
\[
\liminf_{n \to \infty} \frac{\sum_{k=1}^{n} \sum_{l=1}^{n} P(A_k A_l)}{\left( \sum_{k=1}^{n} P(A_k) \right)^2} = 1.
\]

Then there occur with probability 1 infinitely many of the events \( A_n \).

Hence Theorem 2* will follow from

**Lemma 4.** If the event \( A_j \) is defined as
\[
A_j = \{I(N_j, \alpha_2(N_j, T)) < \alpha_2(N_j, T) - T\}
\]
then (1) holds true.
Proof of Lemma 4. Let

\[ B_{ij} = \left\{ \max_{0 \leq k \leq N_i - \alpha_2(N_j, T)} (S_k + \alpha_2(N_j, T) - S_k) < \alpha_2(N_j, T) - T \right\} \]

\( (i < j) \),

\[ C_{ij} = \left\{ \max_{N_i \leq k \leq N_j - \alpha_2(N_j, T)} (S_k + \alpha_2(N_j, T) - S_k) < \alpha_2(N_j, T) - T \right\} \]

\( (i < j) \).

Then

\[ P(A_i A_j) = P(A_i) P(C_{ij}) (1 + o(1)) \]

and

\[ P(A_j) = P(B_{ij}) P(C_{ij}) (1 + o(1)) \]

hence

\[ P(A_i A_j) = \frac{P(A_i) P(A_j)}{P(B_{ij})} (1 + o(1)). \]

By Theorem 5 we also have: \( P(B_{ij}) = 1 + o(1) \) what proves Lemma 4 and Theorem 2 at the same time.

Since

\[ P(S_n - S_{n-a} \geq a - T) = \sum_{j=0}^{T} \binom{a}{j} \frac{1}{2^a} \approx \frac{a^T}{T!} \frac{1}{2^a}. \]

Theorem 4** follows from the Borel – Cantelli Lemma and Theorem 3** is a simple consequence of Lemma A. (To check the conditions of Lemma A is quite easy.)

REFERENCES


P. Erdős – P. Révész