

PROBLEMS IN NUMBER THEORY AND COMBINATORICS

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To the memory of my lifelong friend and collaborator Paul Turán

Last year I gave a talk at this meeting entitled "Problems and results on number theoretic properties of consecutive integers and related problems." (I will refer to this paper as I). I planned to continue with the problems raised in I. Unfortunately, Turán died on September 26, 1976, and I decided to write also a few pages about Turán and our lifelong collaboration and about the effect of Turán's work on my own. I do not plan of course to give a systematic account of Turán's work - this no doubt will be done later. I do not even give a systematic account of our collaboration. First, I restrict myself to number theory and combinatorics, and I choose somewhat arbitrarily, problems which we both investigated (and in some cases initiated) and which can be stated and understood easily.

The paper will consist of three chapters. In the first I discuss Turán's work as it influenced me; in the second I state a few recent problems, some of which originated at this conference; and in the third, I discuss the problems stated in I and give some new related ones.

I

Turán, born in 1910, was 3 years my senior. Both Turán and I worked in the very distant days of our youth in the mathematical journal for high school. (Such a journal existed in Hungary for nearly a century, and it had a great influence in stimulating interest in mathematics.) This journal appeared every month, publishing many problems of various degrees of difficulty. The names of the solvers and also the best solutions were printed. Our first joint work was when such a solution was printed under both our names (we arrived at it independently; we only met at the university, but knew of each other's existence, of course).

I first met Turán in September, 1930. We immediately noticed our common interest in number theory and in prime numbers in

particular. Turán was more interested in analytic methods and I in elementary ones.

In our first joint paper [1] we proved the following theorem: Let $1 \leq a_1 < \dots < a_n$ be any sequence of n integers. Then $v(m)$ is the number of distinct prime factors of m

$$(1) \quad v\left(\prod_{1 \leq i < j \leq n} (a_i + a_j)\right) \geq \frac{\log n}{\log 2} .$$

We always thought that (1) is very far from being the best possible but never could improve it. $\frac{\log n}{\log 2}$ can probably be replaced by $n^{1-\varepsilon}$ and perhaps even by $n/\log n$. If $a_i = i$ we see that (1) cannot hold with $(2 + \varepsilon) \frac{n}{\log n}$. On the other hand we have no counterexample to

$$(2) \quad v\left(\prod_{1 \leq i < j \leq n} (a_i + a_j)\right) \geq (2 + o(1)) \frac{n}{\log n} .$$

Perhaps such a counterexample will not be difficult to find.

It is annoying that we never could settle our conjecture: To every k there is an $f(k)$ so that if $n \geq f(k)$ and $1 \leq a_1 < \dots < a_n$; $1 \leq b_1 < \dots < b_n$ are any two sets of n integers then

$$(3) \quad v\left(\prod_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} (a_i + b_j)\right) > k .$$

I would not be surprised if the proof of (3) is easy and we again overlooked a simple argument.

More generally the following question is perhaps of some interest. Let $G(n)$ be a graph of n vertices x_1, \dots, x_n . Let a_1, \dots, a_n be any set of n distinct integers. Put

$$f(G(n)) = \min v(\Pi(a_i + a_j))$$

where the factor $a_i + a_j$ occurs if and only if x_i and x_j are joined by an edge and the minimum is extended over all choices of distinct

integers a_1, \dots, a_n . It is probably hopeless to determine $f(G(n))$ except for very simple graphs, but perhaps one can get conditions on the class of graphs for which $(f(G(n)) \rightarrow \infty$. One conjecture which just occurs to me states: There is a $g(k) \rightarrow \infty$ as $k \rightarrow \infty$ so that if $G(n)$ has chromatic number $\geq k$ then $f(G(n)) \geq g(k)$. A stronger conjecture: If $G(n)$ has more than kn edges then $f(G(n)) \geq g(k)$. Perhaps this is too optimistic.

In the 1930's, most of our joint work was on interpolation, which I ignore here. One question of Turán had great importance for set theory in general and myself in particular. Turán's problem is stated as follows:

To every real x make correspond a finite set of reals $A(x)$, $x \notin A(x)$. Prove that there is an infinite set G which is independent, i.e., for every $x \in G, A(x) \cap G$ is empty. G. Grünwald and I soon proved Turán's conjecture, but later this problem had many very interesting ramifications in various branches of set theory and combinatorics [2]. It is an interesting fact that Turán was led to his problem by a question of condensation of singularities. As he said, "Fortunately I did not know at that time the classical work which gave sharper results," and thus he was forced to formulate this interesting problem.

Here I want to mention Turán's very simple proof of the result of Hardy and Ramanujan. I and many others often used the methods of this paper [3]. For more details see the forthcoming book of Elliott on probabilistic number theory. Both Turán and I suffered from a regrettable lack of knowledge of probability, thus we missed the Erdős-Kac theorem [3]. In a letter shortly after the war Turán wrote that he realized that he could compute the higher moments of $v(n) - \log \log n$ but refrained from doing this because he saw no applications. The higher moments give a proof of the Erdős-Kac theorem - this method was later used by Delange and Halberstam (see the forthcoming book of Elliott).

I expect that it happens quite often that lack of knowledge prevents a mathematician from proving a theorem which is well within

his reach - e.g., I think Hardy and Littlewood would probably have discovered the law of the iterated logarithm in 1914. This was proved 8 years later by Khintchin [4].

In a later joint paper we investigate the following problem [5]. Let $1 \leq a_1 \leq \dots \leq a_k \leq n$ be a sequence of integers for which the sums $a_i + a_j$ are all distinct. Put $\max k = f(n)$. We proved

$$(1 + o(1))n^{1/2} < f(n) \leq n^{1/2} + n^{1/4}$$

and conjecture

$$(4) \quad f(n) = n^{1/2} + o(1).$$

I offer 500 dollars for a proof or disproof of (4) and 100 dollars for $f(n) \leq n^{1/2} + o(n^{1/4})$. Further we conjectured:

Let $a_1 < a_2 < \dots$ be an infinite sequence of integers. Denote by $g(n)$ the number of solutions of $n = a_i + a_j$. Assume that $g(n) > 0$ for all $n > n_0$. Then

$$(5) \quad \limsup_{n \rightarrow \infty} g(n) = \infty.$$

A slightly sharper form of (5) states: Let $a_k < ck^2$ then (5) holds. I offer 500 dollars for a proof or disproof of these conjectures.

Now I come to Turán's work on extremal problems in graph theory, a subject which was started by him in 1940 and which is now a flourishing branch of graph theory. For the last 30 years I contributed many papers to this subject [6]. Denote by $K^{(r)}(\ell)$ the complete r -graph of ℓ vertices and $\binom{\ell}{r}$ edges. $f(n; K^{(r)}(\ell))$ is the smallest integer for which every $G^{(r)}(n; f(n; K^{(r)}(\ell)))$ (i.e., an r -graph of n vertices and $f(n; K^{(r)}(\ell))$ edges) contains a $K^{(r)}(\ell)$. Turán determined $K^{(2)}(\ell)$ for every ℓ [6] and asked for the determination of $K^{(r)}(\ell)$ for $r > 2$. Almost no progress has been made with this beautiful and intriguing problem. I offer in his memory 500 dollars for the determination of

$$\lim_{n \rightarrow \infty} f(n; K^{(r)}(\ell)) / \binom{n}{r} = c(r, \ell)$$

for any $\ell > r > 2$. Turán also asked for the determination of all the extremal graphs, i.e., all the graphs of $G^{(r)}(n; f(n; K^r(\ell)) - 1)$ which do not contain a $K^{(r)}(\ell)$. Turán completely solved this question for $r = 2$ and every ℓ . He has some plausible conjectures for $r = 3, \ell = 4$ and $r = 3, \ell = 5$. Here are his conjectures. Let $x_1, \dots, x_n; y_1, \dots, y_n; z_1, \dots, z_n$ be the vertices of our graph. The edges are

$$\{x_{i_1}, y_{i_2}, z_{i_3}\}, 1 \leq i_1, i_2, i_3 \leq n \text{ and } \{x_{i_1}, x_{i_2}, y_{i_3}\}, \\ \{y_{i_1}, y_{i_2}, z_{i_3}\}, \{z_{i_1}, z_{i_2}, x_{i_3}\}.$$

If this conjecture is correct then $f^{(3)}(3n; 4, 3) = n^3 + 3n \binom{n}{2} + 1$. Also $f^{(3)}(2n; 5, 3) = n^2(n-1) + 1$ and the extremal graph is $x_1, \dots, x_n; y_1, \dots, y_n$ and its edges are $\{x_i, x_j, y_\ell\}, \{x_\ell, y_i, y_j\}$. If the conjecture is true then the extremal graphs are known not to be unique. The conjecture implies $c(3, 4) = \frac{5}{9}$ and $c(3, 5) = \frac{3}{4}$.

Turán also formulated several other extremal problems on graphs some of which were solved by Gallai and myself [6]. I seriously took up the study of extremal graphs in 1958 and worked on it ever since.

Finally I mention our paper on Turán-Ramsey theorems by V. T. Sós and myself, many of the problems there were solved by Szemerédi, Bollobás and myself [6].

Turán after launching this subject never really returned to it (except his paper with Kővári and V. T. Sós and our joint papers with V. T. Sós (Mrs. Turán) and A. Meir [7]).

To make a big discovery in science or mathematics one certainly needs besides ability and perseverance a certain amount of luck. One has to be there at the right time, but if the opportunity comes (to quote myself) one's "Brain must be open." One of the most famous such cases is the discovery of Roentgen of X-rays. In 1895 Roentgen noticed that if a photographic plate is left near a Crookes tube it gets darkened. He realized that he accidentally observed an important

phenomenon and in a few weeks completed the discovery of his rays which as we all know really changed the world. It seems that Crookes made the same observation several years earlier, but all he deduced was that one should not leave a photographic plate near a Crookes tube.

The reader will, I hope, forgive an old man telling how he missed a more modest but not unimportant discovery. In 1936, I investigated the following problem: How many integers $1 \leq a_1 < \dots < a_k \leq x$ can one give if all the products $a_i a_j$ are distinct. It turned out that

$$(6) \quad \Pi(n) + c_1 n^{3/4} / (\log n)^{3/2} < \max k < \Pi(n) + c_2 n^{3/4} / (\log n)^{3/2}$$

($\Pi(n)$ denotes the number of primes not exceeding n).

To do this I needed to solve the following problem: How many edges must a graph of n vertices have in order to contain a C_4 (a circuit of four edges). I proved without much trouble that $c_3 n^{3/2}$ suffices and suspected that this is best possible. E. Klein (Mrs. Szrkere) soon proved that $c_4 n^{3/2}$ edges do not suffice. Unfortunately, we missed asking more general questions and thus did not start extremal graph theory [8].

Now let me say more of our joint work in number theory. In 1932, we made perhaps our most famous joint conjecture: To every $\epsilon > 0$ and k there is an n_0 so that if $n > n_0$ and $1 \leq a_1 < \dots < a_\ell \leq n$, $\ell > \epsilon n$ then our sequence contains an arithmetic progression of k terms. This problem was open for more than 40 years, and I offered 1000 dollars for a proof or disproof. Finally, early in 1973 Szemerédi proved our conjecture [9]. A more detailed history of the problem can be found in his paper. Recently Fürstenberg proved Szemerédi's theorem by ergodic theory. Fürstenberg's paper will be published soon.

Here I state the following old conjecture of mine: Let $a_1 < a_2 < \dots$ be an infinite sequence of integers satisfying $\sum \frac{1}{a_i} = \infty$. Then our sequence contains arbitrarily long arithmetic progressions. I offer 3000 dollars for a proof or disproof of this conjecture. If true this would imply that for every k there are k primes which form an arithmetic progression of k terms.

This remark will convince the reader that the conjecture will be very hard to settle, and in fact I do not expect to have to pay for it and often said: I should leave some money for it if I leave (the second leave stands of course for death).

In a paper in 1948 [10] we investigate some unconventional problems on primes and state a few interesting conjectures. Put $d_n = p_{n+1} - p_n$. We easily prove that $d_{n+1} > d_n$ and $d_{n+1} < d_n$ both have infinitely many solutions. We tried to prove that $d_{n+2} > d_{n+1} > d_n$ has infinitely many solutions but were unsuccessful. We could not even prove that $d_{n+2} \geq d_{n+1} \geq d_n$ or $d_{n+2} \leq d_{n+1} \leq d_n$ has infinitely many solutions. If this would be false then there is an n_0 so that for every i and $n \geq n_0$, $d_{n+2i} > d_{n+2i+1}$ and $d_{n+2i+1} < d_{n+2i+2}$. It is of course intuitively clear that such an i does not exist, but we could not prove it. I offer 100 dollars for such a proof. Probably $d_n = d_{n+1}$ has infinitely many solutions.

I hardly mention what was perhaps Turán's most important and enduring contribution to mathematics - his "new method in number theory and analysis" [11]. My excuse is that in this paper I deal mainly about our collaboration, and I had little to do with the development of this fruitful method.

For many years (more than 20) Turán worked on the English edition of his German book [11] but in fact this will really be not a translation but an entirely new book - unfortunately it is not completely finished. I very much hope that it will be possible to finish it soon in his spirit. Pressure of other work and also his many new ideas which always forced him to extend and improve his results prevented him from completing his great work. He had many collaborators here. I name some of them: first of all, Knapowski who unfortunately died many years ago, then Tijdeman, Galász, Dancsintz and V. T. Sós = Mrs. Turán. It is certain that they will be able to finish the book in his spirit soon.

Now let me relate my small contributions to the development of his theory. Before 1940 Turán wrote me that the following problem would be important in various branches of mathematics.

Let $1 = |z_1| \geq \dots \geq |z_n|$ be n arbitrary complex numbers:
 Prove that there is an absolute constant c so that

$$(7) \quad \max_{1 \leq k \leq n} \left| \sum_{i=1}^n z_i^k \right| > c .$$

I proved the weaker result $c/\log n$ which for a while was best possible. But my really important contribution to the solution was that in Canberra in 1960 I told the problem to Atkinson [12] who proved (7) with $c = \frac{1}{6}$ which he later improved to $c = \frac{1}{3}$. Turán conjectures that (7) in fact holds for every $c < 1$ if $n > n_0(c)$ is sufficiently large.

I made one more contribution to the theory of power sums [11]. There is an absolute constant $c > 1$ so that for every n there are complex numbers $1 = |z_1| \geq \dots \geq |z_n|$ for which

$$(8) \quad \max_{1 \leq k \leq n+1} \left| \sum_{i=1}^n z_i^k \right| < c^{-n} .$$

The largest value of c for which (12) holds is not known, though my original value has been improved.

I also conjectured that (8) cannot hold if all the $|z_i|$ are ≥ 1 .

Turán restricted himself (due to lack of time) to work only on those power sum inequalities which had applications, and he found very many of them, but I have neither time nor the knowledge to go into more details.

Turán collaborated a great deal with Knapowski on the comparative theory of prime numbers. Very many problems remain in this subject. One of the favorite problems of Turán was: Let $2 = p_1 < p_2 < \dots$ be an infinite sequence of primes. Are there infinitely many values of i for which $p_i \equiv p_{i+1} \equiv p_{i+2} \pmod{4}$? Another favorite problem of his was the prime number race: Denote by $\Pi(n; a, b)$ the number of primes $p < n$, $p \equiv a \pmod{b}$. Is it true that for every $(a, b) = 1$ there are infinitely many values of n for which

$$\Pi(n; a, b) > \Pi(n; a_1, b)$$

for every $a_1 \not\equiv a \pmod{b}$? Special cases were settled by Turán and Knapowski. The general case is far from being settled.

My last mathematical discussion with Turán occurred a few days before his death just before I left for Canada. Put $\epsilon_i = +1$ if $p_i \equiv p_{i+1} \pmod{4}$ and $\epsilon_i = -1$ if $p_i \not\equiv p_{i+1} \pmod{4}$. I am sure that $\sum_{i=1}^n \epsilon_i = o(n)$ and in fact that $\sum_{i=n}^n \epsilon_i = o(n^{1/2+\epsilon})$. Turán completely disagreed and he certainly thought that $\frac{1}{n} \sum_{i=1}^n \epsilon_i$ could very well oscillate between $+1$ and -1 . He was not at all impressed by my heuristic probabilistic arguments. A final conjecture - none alive now will know which of us was right. Incidentally Turán disagreed with this pessimistic view. It was rather sad to hear him say, "it is unfortunate that I get tired so fast and I cannot work very much - I have so many things to finish."

From 1962 up to his death we worked a great deal on the subject which Turán called statistical group theory [13]. We worked in the last few days of his life with a younger mathematician named Szalai on a fairly technical problem. One of our last conversations was about these problems.

Turán found shortly after the war his famous inequality: $-1 \leq x \leq 1$, then $P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \geq 0$ where $P_n(x)$ is the n^{th} Legendre polynomial. Many different proofs and generalizations of this by now classical inequality were published [14]. Turán always felt a bit sorry that this inequality aroused so much more interest than his much deeper power sum method.

Once traveling in Holland he met in the train a mathematician who worked on special functions. Turán introduced himself by writing this inequality and asked, "Do you know this inequality?" "Of course, this is Turán's inequality." "Well, I am Turán."

To end this short history of our collaboration let me state a geometric gem which Turán found in 1932 or early 1933 and which I think was never published. In 1932 E. Klein (Mrs. E. Szekeres) proved that if there are five points in the plane, no three on a line, then one

can always find 4 of them which form the vertices of a convex quadrilateral, and asked to determine the smallest $f(n)$ so that $f(n)$ points contain the vertices of a convex n -gon. Szekeres conjectured $f(n) = 2^{n-1} + 1$, i.e., $f(5) = 9$. Turán and Makai proved this special case. Turán settled the case if the least convex polygon containing the nine points is a quadrilateral. Here is his simple and ingenious proof: Let $\{x_1, x_2, x_3, x_4\}$ be the vertices of the least convex polygon. If the five points inside form a convex polygon there is nothing to prove. If not we can assume that there is a triangle $\{x_5, x_6, x_7\}$ containing say x_8 . The lines $\{x_8 x_5\}$, $\{x_8 x_6\}$, $\{x_8 x_7\}$ divide the plane into three parts, at least one part (say $\{x_5, x_8, x_6\}$) contains two of the points x_1, x_2, x_3, x_4 - say x_1 and x_2 . But then $\{x_1, x_2, x_5, x_8, x_6\}$ forms a convex pentagon! [15].

To sum up Turán not only did pioneering work in his chosen fields of analytic number theory and various branches of analysis but he also had the remarkable ability of initiating new directions of research in various branches of mathematics in which he only had a passing interest.

Perhaps the reader will permit a very old man whose cure* is perhaps not very far to make some personal reminiscences.

We first met in the mathematical seminar room early in September of 1930. I asked him, "Is it true that the sum of reciprocals of the primes diverges?" Turán informed me that this was proved by Euler and also told me about the prime number theorem.

We talked a great deal of mathematics in the 1930's. I did not always explain my proofs very clearly and once we argued heatedly. I thought that he was too slow, but he floored me completely by saying, "I am sure that you are wrong - mathematics can be understood when clearly presented, and I am not so very stupid as not to understand a clearly presented argument." Incidentally, I remember now very clearly when I told him early in December, 1933, the proof that the sum of the

*Cure = death, i.e., the cure of the incurable disease of life.

reciprocals of the primitive abundant numbers converges.

In the early 1930's Turán used to say, "There are three things which make life interesting: Mathematics, Women and Music - in this order of importance." He maintained his interest in all three of these things - I just proved this about mathematics. There was a radio at his bedside in the hospital, and he often listened to music. Finally, we talked about a common friend from abroad who got captured (=married) more than 10 years ago. He said, "As you know, not much is needed that I should turn around to look at a boss (= female; slave = male), but I would not turn around to look at his boss."

Turán in the distant days of his youth believed that a pretty girl cannot be interested in mathematics. It was told: He married the counterexample.*

In our youth we often talked about Mathematics while walking, i.e., we proved and conjectured without paper - later in life Turán did not like this anymore and preferred to have proofs written down to be able to check them. In the 1930's it often happened that when we talked about mathematics while walking on the main streets of Budapest he suggested, "Let us walk on a side street - there are too many pretty girls here to distract me." One further anecdote: In a paper on child prodigies I write about Lovász and tell the following true story about him: He went to a special high school for mathematically gifted young people. When he was not quite 15, he and one of his friends in this school courted a girl who also studied there. They asked her to choose between them and she chose Lovász. They have two ϵ 's now (ϵ = child), and she is also a professional mathematician. When I told this story to E. Milner he embellished it by adding: she of course said, "I will choose the one who proves the Riemann hypothesis." In 500 years the story would have perhaps been remembered in this way, but Turán further improved it by adding in happier days, "I did not know that she wants to marry me," i.e., of course, he will prove it himself.

*I don't mean to alienate the boss (= female) class by saying "the" counterexample rather than "a" counterexample, but this is the way the joke was told.

Turán shared with me a great interest in history. He had an amusing weakness. He was very fond of making puns (not necessarily good ones) in every possible language. Turán was also very interested in sports both as a spectator and a participant. Before getting ill he enjoyed playing soccer and table tennis.

One of his nice sayings was one which he once wrote me to console me, "A strong fortress is for us our Mathematics." Renyi used to say, "If I am in a good mood I do mathematics to get in a still better mood - if I am in a bad mood I do mathematics that my blues should disappear." Turán shared this rare ability with Renyi.

It is always sad if a great man dies when he is still in his prime mentally, but there is a consolation: He never knew the two greatest evils - old age and stupidity. I hope I too will be spared them as Turán was.

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P. Erdős and V. T. Sós, *Some remarks on Ramsey's and Turán's theorem*, Colloq. Math. Soc. J. Bolyai 4, Combinatorial theory and its applications II (1970), 393-405; E. Szemerédi, *Graphs without a complete quadrilateral*, Mat. Lapok 23 (1973), 113-116 (in Hungarian); B. Bollobás and Paul Erdős, *On a Ramsey-Turán type problem*, J. Comb. Theory 21 (1976), 166-168.

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II

I now state a few very recent problems on combinatorics and number theory some of which occurred to me during or immediately before our meeting.

Let $|S| = n$ be a set $2 \leq |A_k| < n$, $1 \leq k \leq m$ is a family of subsets of S so that every pair (x,y) of elements of S is contained in one and only one of the A 's. A theorem of de Bruijn and myself states that $m \geq n$ [1]. Let now $F(n)$ be the smallest integer for which there is a system $\{A_k\}$ with the above properties so that for every r ($2 \leq r \leq n - 1$) the number of indices i with $|A_i| = r$ is at most $F(n)$. It would be interesting to determine $F(n)$, but perhaps there is no simple formula for it. I conjecture

$$(1) \quad F(n) = cn^{1/2} + o(n^{1/2}) .$$

It is a simple consequence of our theorem with de Bruijn that $F(n) > c_1 n^{1/2}$, but I could not prove $F(n) < c_2 n^{1/2}$. Perhaps this will not be so very difficult but I am inexperienced in constructing designs.

It would be interesting to determine or estimate the number $A(n)$ of sequences

$$(2) \quad \{\alpha_1, \alpha_2, \dots, \alpha_m\}, |\alpha_1| \geq \dots \geq |\alpha_m| \geq 2$$

for which there is a system $A_k \subset S$, $|S| = m$, $|A_k| = \alpha_k$ so that every pair (x,y) of S is contained in one and only one A_k . The difficulty is that we do not have any necessary and sufficient condition for a sequence (2) which would ensure the existence of such a set system.

Assume now that the x_i 's are points in the plane and the A 's are the lines determined by the points. Purdy and I just proved that in this case $F(n) = n - 1$ if $n > n_0$, i.e., there is an r so that there are at least $n - 1$ lines which contain exactly r points. In fact $r = 2$ or $r = 3$. Our proof will be published elsewhere.

Denote by $B(n)$ the number of sequences

$$\{\alpha_1, \dots, \alpha_m\}, |\alpha_1| \geq \dots \geq |\alpha_m| \geq 2$$

for which there is a set of n points in the plane which determine the lines L_1, \dots, L_m , L_i contains exactly α_i points. I am sure that $B(n)$ is much smaller than $A(n)$.

A problem of Mullin led me to the following question: $f(n;k)$ is the smallest integer for which there is a graph $G(n;f(n;k))$ (i.e., a graph of n vertices and $f(n;k)$ edges) so that if G_1 and G_2 are two graphs which are isomorphic to our G and which have the same vertices then G_1 and G_2 have at least k edges in common. Determine or estimate $f(n;k)$. We obtain by a simple averaging process

$$k^{1/2}n - c \leq f(n;k) \leq 4k^{1/2}n$$

and for large k ($k > n^E$) $f(n;k) = (2 + o(1))k^{1/2}n$.

Perhaps it is more interesting to determine $f(n;k)$ for small values of k . $f(n;1) = n - 1$, $f(n;2) = f(n;3) = \lfloor 3n/2 \rfloor$ was proved by Chung, Graham and Murty. $f(n;4) = 2n - 4$? $f(n;4) \leq 2n - 4$ is immediate. Consider the complete bipartite graph of two white and $n - 2$ black vertices. An example of Mullin shows that perhaps $f(n;5) = 2n - 2$. Let our G have a cycle of length $n - 1$ and the n^{th} vertex be joined to all vertices of the cycle. Mullin observed that this graph gives $f(n;5) \leq 2n - 2$. I have no idea about $f(n;k)$ for $k > 5$.

Graham just telephoned me (1976 \bar{X} 11), Mrs. Chung and he conjecture that

$$(3) \quad f\left(n; \left\lfloor \frac{t(t-1)}{4} \right\rfloor\right) = \frac{tn}{2} + o(n).$$

and that perhaps (3) can be strengthened to $\frac{tn}{2} + o(1)$.

Put $s_1(n) = \sigma(n) - n$ where $\sigma(n)$ is the sum of divisors of n and $s_{k+1}(n) = s_1(s_k(n))$. A well-known conjecture of Catalan states that the sequence $\{s_k(n)\}$, $k = 1, 2, \dots$ is bounded for every n . This question was investigated recently by the Lehmers, Guy and Selfridge and others. Guy and Selfridge have good heuristic arguments which seem to indicate that Catalan's conjecture is false for almost all even numbers [2]. Pomerance

and Penney decided to investigate a simpler function: Put

$$n = \prod_i p_i^{\alpha_i}, \quad \bar{\sigma}(n) = \prod_i p_i^{\alpha_i - 1} (p_i + 1), \quad f_1(n) = \bar{\sigma}(n) - n, \quad f_k(n) = f_1(f_{k-1}(n)).$$

They proved that the smallest integer for which $f_k(n) \rightarrow \infty$ is $n = 318 = 6.53$. They continue their investigations. I think it will be hard to characterize the integers n for which $f_k(n) \rightarrow \infty$. Selfridge and I tried to find a function about whose iterates one can really prove something non trivial, but we were not very successful. We considered $h_1(n) = \frac{\sigma(n)}{(\sigma(n), n)}$, $h_k(n) = h_1(h_{k-1}(n))$. Is it true that $\{h_k(n)\}$, $k = 1, 2, \dots$ is bounded for every n ? This conjecture seems very plausible but unfortunately we could prove nothing. We found three cycles: $h_1(8) = 15$, $h_1(15) = 8$, $h_1(512) = 1023$, $h_1(1023) = 512$ and a longer cycle starting with 127.* Presumably there are longer cycles some of which we hope to find. The degree of our ignorance can be illustrated by the fact that we could not prove that for every n there is a k with $h_k(n) > h_{k+1}(n)$ (i.e. the sequence $h_k(n)$, $k = 1, 2, \dots$ can not tend monotonically to infinity). Observe that $\sigma(20^2) = 31^2$. We could not prove that $\sigma(u^2) = v^2$ has only a finite number of solutions and what is even more ridiculous, we could not prove that there does not exist an infinite sequence $n_1 < n_2 < \dots$ with $\sigma(n_i^2) = h_1(n_i^2) = n_{i+1}^2$, $i = 1, 2, \dots$. We did not have time to think of this problem seriously and perhaps we overlooked a trivial idea. (S. Weintraub found many more solutions of $\sigma(n^2) = m^2$ 9^2 , 1306^2 and 1910^2).

On the other hand we could not prove that for every r there is an n_r for which $h_{\ell+1}(n) > h_\ell(n)$ for every $\ell \leq r$.

It follows from results on the distribution of primes in arithmetic progressions without too much difficulty that for almost all n , $h_1(n) = o(n)$, and with a little more trouble it follows from a result of mine [3] for squarefree n

$$(4) \quad \limsup_{n \rightarrow \infty} h_1(n)/n \log \log n = 1.$$

* Penney checked all n up to 10^6 , he found one more cycle of length 2 $n = 29127 = 3.7.19.73$.

Pomerance remarked that if n is not squarefree then $h_1(n)$ can increase just as fast as $\sigma(n)$ i.e.

$$\limsup_{n \rightarrow \infty} h_1(n)/n \log \log n = e^{\gamma}.$$

Selfridge and I classified the primes as follows: In the first class are the primes of the form $2^{\alpha}3^{\beta} - 1$. Assume that the first $r - 1$ classes are already defined. A prime p is in the r^{th} class if all prime factors of $p + 1$ have class $\leq r - 1$ with equality for at least one prime factor.

It seems certain that there are infinitely many primes in all the classes, but we could prove nothing and in fact the problem seems hopeless.

It is easy to prove that the number of primes not exceeding n of the r^{th} class is $o(n^{\epsilon})$ for every $\epsilon > 0$ and r . In fact it would be easy to get a sharper result.

Denote by $p_1^{(r)}$ the least prime of the r^{th} class.

$p_1^{(1)} = 2$, $p_1^{(2)} = 13$, $p_1^{(3)} = 37$, $p_1^{(4)} = 73$. Selfridge and I disagreed on the growth of $p_1^{(r)}$. I conjectured $(p_1^{(r)})^{1/r} \rightarrow \infty$, but Selfridge thought it quite likely that it is bounded. We will never know the truth - at least not in this world.

Benkoski calls an integer n weird if it is abundant (i.e. $\sigma(n) \geq 2n$) and if n is not the distinct sum of proper divisors of n . 70 is the smallest weird number. Benkoski asks: Is there an odd weird number? In a recent joint paper we investigated some properties of weird numbers. Our most interesting open problem states: Is there a constant C so that no integer n with $\frac{\sigma(n)}{n} > C$ is weird? If the answer is affirmative, determine the smallest admissible C [4].

Perhaps the following problem on divisors is of some interest: Let n be an integer. $f(n)$ is the smallest integer for which for

some k $n = \sum_{i=1}^k d_i$ where $1 = d_1 < \dots < d_k \leq f(n)$ are the

consecutive divisors of $f(n)$. e.g. $f(10) = 12$ $10 = 1 + 2 + 3 + 4$,
 $f(7) = 4$, $7 = 1 + 2 + 4$. Is it true that $f(n) = o(n)$? Perhaps this
is too optimistic and $f(n) = o(n)$ holds only for almost all n . One
could perhaps modify the definition of $f(n)$ by permitting representations
of the form $n = \sum_{i=u}^v d_i$ where $1 \leq u \leq v \leq \ell \leq d(f(n))$.

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III

First I review and correct some of the statements and conjectures made in I and then state some related and unrelated problems and theorems.

To make this paper independent of I I will repeat the definitions stated in I as the need arises.

$P(n)$ denotes the greatest and $p(n)$ the least prime factor of n . I stated in I that I can not prove that for every $\epsilon > 0$ there are infinitely many integers n for which

$$(1) \quad P(n) < n^\epsilon, \quad P(n+1) < n^\epsilon.$$

Schinzel pointed it out to me that the proof of (1) is in fact simple. Here is the outline of his proof:

Let $\varphi(m) < \eta m$ where $\eta = \eta(\epsilon)$ is sufficiently small. Put $n = 2^m - 1$ and use

$$2^m - 1 = \prod_{d|m} \phi_d(2)$$

where $\phi_d(x)$ is the d^{th} cyclotomic polynomial. Now $2^m - 1$ and 2^m satisfies (1). Schinzel's proof of course does not give that the density of integers satisfying (1) is positive.

I stated on p.33 of I the conjecture that

$$u_i^{(4)} + u_j^{(4)} = u_l^{(4)}$$

has only a finite number of solutions where $u_i^{(k)}$ is an integer all whose prime factors occur with an exponent $\geq k$. Schinzel writes that the identity

$$(1 + k^n)^n + (k(1 + k^n))^n = (1 + k^n)^{n+1}$$

shows that $(u_i, u_j, u_l) = 1$ has certainly to be assumed. A forthcoming paper of A. Ivic (On asymptotic formulas for powerful numbers, Journal of Number Theory) contains many results on additive properties of powerful numbers.

Put $A(n ; k) = \prod_{i=1}^k (n + i)$. I proved in I that the upper density

of the integers n for which

$$P(A(n ; k)) < n^{1/2-\epsilon}$$

is for $k > k_0(\epsilon, \eta)$ less than η . I stated that there is no doubt that the result remains true if $n^{1/2-\epsilon}$ is replaced by $n^{1-\epsilon}$. I am still sure that this result is true, but was quite surprised that I could not even prove this for $P(A(n ; k)) < n^{1/2}$. Perhaps I overlooked a trivial idea.

Another problem in I stated that for $k \geq 3$ there are only a finite number of values of n, m and $k, m \geq n + k$ for which $A(n ; k)$ and $A(m ; k)$ have the same prime factors. I have no idea how to settle this question. All I can do is to formulate a new conjecture. Denote by $B(n ; k)$ the least common multiple of $n+1, n+2, \dots, n+k$. Is it true that for $m \geq n + k$ $B(n ; k) \neq B(m ; k)$? In fact, probably

$$B(n ; k) = B(m ; k), \quad k > 1, m \geq n + k$$

has only a finite number of solutions.

By the way, many further cases were found by S. Weintraub and A. R. Reddy of integers n, m and $k, m \geq n + k$ where $A(n ; k)$ and $A(m ; k)$ have the same prime factors, no solutions so far for $k > 5$. Reddy even found $n_1, n_2, n_3, n_2 \geq n_1 + 3, n_3 \geq n_2 + 3$ so that $A(n_1 ; 3), A(n_2 ; 3)$ and $A(n_3 ; 3)$ all have the same prime factors.

Put $(p^\alpha \parallel m)$ denotes $p^\alpha | m, p^{\alpha+1} \nmid m$

$$S(m ; k) = \prod p_i^{\alpha_i} \quad \text{where } p_i^{\alpha_i} \parallel m, p_i \leq k.$$

In I I stated (in slightly different notation) that there is a constant C so that to every $\epsilon > 0$ there is a k_0 for which for every fixed $k > k_0$ the density of integers n for which

$$(2) \quad C^k(1 - \epsilon)^k < S\left(\binom{n}{k}, k\right) < C^k(1 + \epsilon)^k$$

is greater than $1 - \epsilon$. Can $(1 + \epsilon)^k$ be replaced in (2) by $e^{k^{1/2+\epsilon}}$? Is it true that

$$k^{-1/2} \log(S(\binom{n}{k}, k) - Ck)$$

has a distribution function? Perhaps this question can be settled without too much difficulty by the method of moments.

Let n be an integer. Denote by $t_k(n)$ the smallest integer for which

$$\prod_{i=1}^k (t_k(n) + i) \equiv 0 \pmod{n}.$$

I conjectured and R. R. Hall proved (unpublished) that

$$\sum_{n=1}^x t_2(n) = o(x^2).$$

It would be of some interest to obtain an asymptotic formula for

$$\sum_{n=1}^x t_2(n)$$

and to estimate as accurately as possible the value of $t_2(n)$

for almost all n .

Let $k(n)$ be the smallest integer r for which $r! \equiv 0 \pmod{n}$. In other words r is the smallest integer for which $t_{r-1}(n) = 1$. Trivially

$$n-1 = t_1(n) > t_2(n) > \dots > t_{r-1}(n) = 1 \quad (\text{since } t_k(n) \leq t_{k-1}(n) - 1).$$

It would interest me to prove that for infinitely many n

$$(3) \quad t_{k+1}(n) < t_k(n) - 1, \quad 1 \leq k < r.$$

In I I conjectured that perhaps (3) holds for every $n = m!$ if $m > m_0$. Perhaps it is not quite hopeless to characterize the integers n for which (3) holds, or at least find usable necessary and usable sufficient conditions for n to satisfy (3). It seems obvious that for "small i " $t_{i+1}(n) < t_i(n) - 1$ for almost all n , perhaps one can obtain some more quantitative results here. Put

$$\alpha_j(n) = \frac{1}{n} \prod_{i=1}^j (t_j(n) + i), \quad 1 \leq j \leq r = k(n).$$

It immediately follows from the theorem of Thue and Siegel that $\max(\alpha_2(n), \alpha_3(n))$ tends to infinity as n tends to infinity. How fast? Is it greater than n^ϵ for some fixed $\epsilon > 0$?

Put further

$$F(n) = \max_{1 \leq i \leq r} \alpha_i(n).$$

Probably $F(n)$ tends to infinity very fast. The integers $\alpha_i(n)$, $1 \leq i \leq r$, are perhaps all distinct for $n > n_0$.

By the way it is easy to see that for almost all n , $k(n) = P(n)$ and it is not very much harder to show that the number of integers $n < x$ with $k(n) > P(n)$ is for a certain $r > 0$ equal to $x \exp(-c + o(1)) (\log x \log \log x)^{1/2}$.

Finally can one characterize in any way a set of primes $2 = p_1 < \dots < p_k$ for which there is an n so that all prime factors of $n(n+1)$ are amongst the p 's². I would guess that for fixed k there are relatively few such k -tuples, but as far as I know nothing is known here. The character of the problem changes if we further require that all the primes p_1, \dots, p_k actually occur in $n(n+1)$. Ecklund conjectured that if the p 's are the first k primes and $p_k > 29$ then such an n does not exist.