A Class of Hamiltonian Regular Graphs

Paul Erdös
HUNGARIAN ACADEMY OF SCIENCES

Arthur M. Hobbs
TEXAS A & M UNIVERSITY

ABSTRACT

In this paper, we show that if $n \geq 4$ and if G is a 2-connected graph with $2n$ or $2n-1$ vertices which is regular of degree $n-2$, then G is Hamiltonian if and only if G is not the Petersen graph.

We use the terminology of Behzad and Chartrand [2]. In particular, a set of vertices in a graph is *independent* if no two of the vertices in the set are adjacent. A graph is *cubic* if every vertex of the graph has degree three.

Dirac [6] showed that if G is a graph with $m \geq 3$ vertices and if every vertex of G has degree $\frac{1}{2}m$ or more, then G is Hamiltonian. Dirac's work has been extended in [10], [11], [3], [5], [8], and [4], but these results all require the existence of vertices of degree at least $\frac{1}{2}m$. Avoiding this latter requirement, Gordon [7] recently proved the following:

Theorem. Let G be a finite graph with $2n$ vertices in which every vertex has degree at least $n-1$. Then either G is Hamiltonian, G has a subgraph isomorphic to $K_{n+1,n-1}$, G has a subgraph isomorphic to $G_{2n,b}$ for some $b \leq n$, or G has a subgraph isomorphic to H, where $G_{2n,b}$ and H are precisely defined non-Hamiltonian graphs.

As a consequence of Gordon's theorem, if $n \geq 3$ and if G is a 2-connected graph with $2n$ vertices which is regular of degree $n-1$, then G is Hamiltonian.

Journal of Graph Theory, Vol. 2 (1978) 129–135
 0364–9024/78/0002–0129$01.00

We will need the following three theorems:

Theorem A (Dirac [6]). Let G be a 2-connected graph with m vertices in which every vertex has degree k or more. Then either $m<2k$ or G includes a cycle of length $2k$ or more.

Theorem B (Moon and Moser [9]). Let $n\geq 2$. If $B(n, n)$ is a bipartite graph with n vertices in each color class, and if every vertex in $B(n, n)$ has degree greater than $\frac{1}{2}n$, then $B(n, n)$ is Hamiltonian.

Theorem C (Derived from Balaban et al. [1]). the Petersen graph is the only cubic block with at most ten vertices which is not Hamiltonian.

While every Hamiltonian graph is 2-connected, it is not always necessary to include this property as a condition in a theorem whose conclusion is that a class of graphs is Hamiltonian (e.g., Dirac's 1952 theorem). However, in the case of the theorem proved here, 2-connectedness must be required, as is shown by the class of graphs described below. Given a function f which assigns a non-negative integer to each vertex of a graph G, an f-factor of G is a spanning subgraph S of G such that the degree of each vertex u in S is $f(u)$. It is not difficult to show the following theorem:

Theorem 1. Let G be a graph with $2n-m$, $m\in\{0, 1\}$, vertices which is regular of degree $n-2$. Then G is not 2-connected if and only if there are subgraphs F and H of G such that $G=F\cup H$, there is a vertex v with $V(F\cap H)=\{v\}$, and there is an integer p such that $2\leq 2p\leq n$, F is formed from K_{n+1-m} by removing the edges of an f-factor of K_{n+1-m} with $f(v)=2p-m$ and $f(u)=2-m$ for all u in $V(k_{n+1-m})-\{v\}$, and H is formed from K_n by removing the edges of an h-factor of K_n with $h(v)=n-2p+1$ and $h(u)=1$ for all u in $V(K_n)-\{v\}$.

In this theorem, note that $F\cup H$ is not connected if $2p\in\{2, n\}$, and $F\cup H$ has a bridge if $2p=n-1$.

Throughout the remainder of this paper, n is a positive integer, G is a 2-connected graph with $2n$ or $2n-1$ vertices which is regular of degree $n-2$, P is a cycle of maximum length in G, $R=V(G)-V(P)$, r is the number of elements of R, and v and w are used only to name vertices in R. Further, given $v\in R$ and given a direction around P, C is the set of vertices of P adjacent to v, A is the set of vertices immediately preceding vertices of C on P, and B is the set of vertices immediately following vertices C on P. It is easily seen that

OBSERVATION: $A \cup \{v\}$ and $B \cup \{v\}$ are independent sets of vertices.

In the proof of Theorem 2, we first show that R is independent. Using the independence of R, one can easily show that $r \leq 1$. Finally, we examine the remaining case of $r = 1$ and find only the Petersen graph is not Hamiltonian.

Lemma 1. Suppose v and w are in R, $v \neq w$, and suppose v and w are joined by a path of length k in $G - V(P)$. If v is joined to a vertex c of P and w to a different vertex c' of P, then between c and c' on P there are at least $k+1$ vertices not adjacent to either v or w.

Proof. If the lemma fails, we may suppose that between c and c' there are k or fewer vertices joined to neither v nor w and no vertices joined to either v or w. But then a longer cycle than P can be formed by replacing the portion of P from c to c' by the path of length k joining v and w together with the edges from v and w to c and c'. Thus the lemma is true. ∎

Lemma 2. Let v and w be distinct vertices of a component S of $G - V(P)$, and suppose there is a path of length k in S joining v and w. Suppose the number of edges from v and w to vertices of P is j and suppose that, going around P, there are i cases in which a vertex of P which is joined to exactly one of v or w is followed by a vertex joined to the other of v and w with no vertices joined to either between them and i' cases in which a vertex of P is joined to both v and w. Then

$$2n - m - (k+1) \geq |V(P)| \geq j + ik + i'(k-1) + (j - i').$$

Proof. The upper bound is obvious. Since there are $j - i'$ vertices of P joined to v and/or w, it is sufficient to show that P has at least $j + ik + i'(k-1)$ vertices not joined to either v or w. Suppose v and w are both joined to a vertex c of P. Then between c and the next vertex c' of P joined to either of v or w there are at least $k+1$ vertices joined to neither. Allowing for two of these to be counted against the edges joining v and w to c, there are $k-1$ vertices between c and c' which are not counted against edges. If a vertex c of P is joined to either of v or w but not both, then the next vertex on P is joined to neither by Lemma 1 and the observation and can be counted against the edge to c; further if the next vertex c' after c on P which is joined to either v or w is joined to the one of these not joined to c, there are still k of the vertices between c and c' which are not counted against any edge from v or w to P. The lemma follows. ∎

In the remainder of this paper, the symbols i, i', j, and k are as defined in Lemma 2.

Lemma 3. If $G-V(P)$ has a nontrivial component S, then $n \leq 4$.

Proof. Let d be the smallest degree in S for which there are two vertices of S of degree d in S. Since S has no more than four vertices but is nontrivial, S has at least two vertices of degree exactly $d \in \{1, 2, 3\}$ joined by a path of length at least d. Then Lemma 2 together with $j = 2(n-2-d)$ and $j - i' \geq n-2-d$ imply that

$$2n-d-1-m \geq 3(n-2-d)+id+i'(d-1), \tag{1}$$

which yields

$$0 \geq (n-5)+d(i-2)+i'(d-1)+m. \tag{2}$$

If $i \neq 0$, then i is at least two, so n is no more than 5 by (2). Thus $i=2$ and $n=5$. Now we have $0 \geq i'(d-1)+m$, so $m=0$. Since $n=5$, G is cubic, so $|V(P)| \geq |V(G)|-1$ by Theorem C. Thus $i=0$.

Now from the definition of i, if there is a vertex of P joined to just one of v and w, then no vertex of P can be joined to the other of v or w. Thus $i' = j/2 = n-2-d$. Substituting this into (1), we obtain

$$d^2+3d+3-m \geq dn. \tag{3}$$

If $i'=0$, then $n=5$ and $d=3$; but this case is finished. Thus $i'>0$ and $n \geq 6$. We now consider the three choices for d.

Case 1. Let $d=1$ and note that $n \in \{6, 7\}$ by (3). If $n=7$, then $i'=4$, so P has at least 12 vertices by Lemma 1. Thus $|V(S)|=2$ and each vertex of S is adjacent to the same equally spaced $i'=4$ vertices in P. If any vertex in A is adjacent to more than one vertex in B, then G has a cycle longer than P. Thus each vertex in A is adjacent to every vertex in C and this implies that the vertices of C have degree 6 or more, which is impossible. Thus $n \neq 7$. If $n=6$, then $i'=3$ and P has either 9 or 10 vertices by Lemma 1. In either case, P has a subpath c, b, a, c' where c and c' are in C. If b is adjacent to at least two vertices in A, then G has a cycle longer than P. Thus b is adjacent to at least two vertices in C, so C has a vertex with degree at least 5. Since this is impossible, $d \neq 1$.

Case 2. Let $d=2$ and note that $n=6$. Also $i'=2$, so P has either 8 or 9 vertices by Lemma 1. Since S does not have two vertices of degree 1, S

must have a cycle containing the two vertices v and w. Let x be another vertex on the cycle in S. Since x can be joined to at most 3 vertices of S, x must be joined to a vertex of P. But x cannot be joined to a member of C, and there is a path in S between x and each of v and w of length at least two. Thus a longer cycle than P exists in G no matter what vertex of P is joined to x. Thus $d \neq 2$.

Case 3. Let $d=3$ and note that $n=6$ or $n=7$. Clearly, $i'=n-d-2 \geq 1$. Also, $S=K_4$ and this, by symmetry, implies that each vertex of S is adjacent to the same i' vertices in P. Thus G has a vertex (in C) which has degree at least 6. Since this is impossible, the result follows. ∎

Lemma 4. If $n \geq 5$, then R has order $r \leq 1$.

Proof. If $\langle \{w\} \cup A \rangle$ has at least two edges, then w is adjacent to two vertices of A and we can easily find a cycle longer than P. Thus $\langle \{w\} \cup A \rangle$ has no more than one edge. Similarly, $\langle \{w\} \cup B \rangle$ has no more than one edge.

Let p be the number of edges in $\langle A \cup R \rangle$. Since $\{v\}$ together with A is an independent set, $p \leq r-1$ and G has exactly $(n-2+r)(n-2)-2p$ edges joining vertices in $A \cup R$ to those in $V(G)-(A \cup R)$. Thus

$$|E(G)| = n(n-2) - m(n-2)/2 \geq (n-2+r)(n-2) - 2p + p,$$

and this implies that $(r-2+m/2)(n-3) \leq 1 - m/2$.

Since $n \geq 5$, we have $r \leq 5/2 - 3m/4$. Thus, if $m=1$, $r \leq 1$. Suppose $m=0$, so $r \leq 2$. Supposing that $r=2$, note that $p \leq r-1=1$. If $p=0$, then $A \cup R$ is an independent set of n vertices and Theorem B implies that G is Hamiltonian. Thus $\langle A \cup R \rangle$ has n vertices and exactly one edge and, therefore, $G-(A \cup R)$ has exactly one edge. Thus G is Hamiltonian by Theorem B if $n \geq 6$. If $n=5$, the lemma follows from Theorem C. ∎

Let $D = A \cap B$ and let $X = V(P) - (A \cup B \cup C)$.

Lemma 5. If $n \geq 5$, then $X = \phi$.

Proof. Clearly $|X| \leq 2-m$. Suppose that $X=\{x, y\}$ and note that x and y are consecutive in P. Thus P contains the subpath c, b, x, y, a, c' where $\{a\} = A-B$ and $\{b\} = B-A$. If both x and y are adjacent to vertices in D, then we can easily find a Hamiltonian cycle in G. If x is adjacent to no vertex of D, then $\langle B \cup \{v, x\} \rangle = H$ has n vertices and one edge and this implies that $G - V(H)$ has one edge. However, $G - V(H)$ has the subpath y, a, c', which is a contradiction. Likewise if y is adjacent to no vertex of D, then $\langle A \cup \{v, y\}\rangle$ has only one edge and this leads to a contradiction. Thus $|X| \neq 2$.

Suppose that $X = \{x\}$. If $m = 0$, note that $|A - B| = |B - A| = 2$. Let $A - B = \{a, a'\}$ and $B - A = \{b, b'\}$, where c, b, x, a, c' and b', a' are subpaths of P. G is Hamiltonian if edges ab' and $a'b$ are both in G. If ab' is not an edge of G, then $\langle A \cup \{v, b'\}\rangle = H$ has n vertices and one edge while $G - V(H)$ contains xb and bc; this is impossible. Likewise, if $a'b$ is not an edge, then $\langle B \cup \{v, a'\}\rangle = H'$ has n vertices and one edge while $G - V(H')$ has edges xa and ac'; again this is a contradiction. Thus $|X| \neq 1$ if $m = 0$. If $m = 1$, then $\langle A \cup B \cup \{v\}\rangle$ has n vertices and at most one edge. If $\langle A \cup B \cup \{v\}\rangle$ has no edge, then $n(n-2) \leq (n-1)(n-2)$, which is impossible. Otherwise, $n(n-2) - 2 \leq (n-1)(n-2)$, whence $n \leq 4$. Thus $X = \phi$.

Theorem 2. If $n \geq 4$ and G is a 2-connected $(n-2)$-regular graph with $2n$ or $2n-1$ vertices, then either G is Hamiltonian or G is the Petersen graph.

Proof. If G is not Hamiltonian, then $n \geqslant 5$ and $V(P) = A \cup B \cup C$. Also, $|V(P)| = 2n - 1 - m$ and $|A - B| = |B - A| = 3 - m$. Let $A - B = \{a_1, \ldots, a_{3-m}\}$ and let $B - A = \{b_1, \ldots, b_{3-m}\}$, where $a_i b_i$ are edges of P for $i = 1, \ldots, 3-m$ and they occur in cyclic order on P.

Suppose $m = 0$. If $\langle (A \cup B) - D\rangle = H$ has at least seven edges, then we can easily find a Hamiltonian cycle in G using edges $a_i b_j$ and $a_j b_i$ for some $i \neq j$. Thus H has no more than six edges, and this implies that $\langle A \cup B \cup \{v\}\rangle$ has $n+2$ vertices and no more than six edges. Thus $(n+2)(n-2) - 12 \leq (n-2)(n-2)$, or $n \leq 5$. But if $n = 5$, the theorem follows from Theorem C. The case $m = 1$ is similar.

If $n = 4$, G is a 2-connected 2-regular graph, i.e., G is a cycle. The theorem follows. ∎

ACKNOWLEDGMENT

The authors wish to thank the referee for substantial suggestions which significantly reduced the length of this paper.

References

[1] A. T. Balaban, R. O. Davies, F. Harary, A. Hill, and R. Westwick, Cubic identity graphs and planar graphs derived from trees. *J. Austral. Math. Soc.* 11 (1970) 207–215.

[2] M. Behzad and G. Chartrand, *Introduction to the Theory of Graphs.* Allyn and Bacon, Boston (1971).

[3] J. A. Bondy, Properties of graphs with constraints on degrees. *Studia Sci. Math. Hung.* 4 (1969) 473–475.

[4] J. A. Bondy and V. Chvátal, A method in graph theory. *Discrete Math.* 15 (1976) 111–135.

[5] V. Chvátal, On Hamilton's ideals. *J. Combinatorial Theory* 12B (1972) 163–168.

[6] G. A. Dirac, Some theorems on abstract graphs. *Proc. London Math. Soc., Ser.* 3, 2 (1952) 69–81.

[7] L. Gordon, Hamiltonian circuits in graphs with many edges. Unpublished report, Sydney University, Australia.

[8] M. Las Vergnas, Sur une propriété des arbres maximaux dans un graphe. *C. R. Acad. Sci. Paris* 272 (1971) 1297–1300.

[9] J. Moon and L. Moser, On Hamiltonian bipartite graphs. *Israel J. Math.* 1 (1963) 163–165.

[10] O. Ore, Note on Hamiltonian circuits. *Amer. Math. Monthly* 67 (1960) 55.

[11] L. Pósa, A theorem concerning Hamiltonian lines. *Maygar Tud. Akad. Mat. Kutató Inst. Közl.* 7 (1962) 225–226.