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ON FINITE SUPERUNIVERSAL GRAPHS

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Define a simple graph G to be k-*superuniversal* iff for any k-element simple graph K and for any full subgraph H of K every full embedding of H into G can be extended to a full embedding of K into G. We prove that for each positive integer k there exist finite k-superuniversal graphs, and we find upper and lower bounds on the smallest such graphs. We also find various bounds on the number of edges as well as the maximal and minimal valence of a k-superuniversal graph. We then generalize the notion of k-superuniversality to cover graphs with colorings and prove similar and related theorems.

1. Introduction

In this paper we shall consider a generalization of universality and its application to finite graphs. Consider the graphs F, G, and H, below (Fig. 1). Clearly F is universal with respect to 3-element simple graphs in that any such graph can be fully embedded into it. On the other hand, if we take the graph G and begin to embed it into F by mapping r to a and s to b, then we cannot complete the embedding. However, it can be seen that if we embed any part of any 3-element simple graph into the graph H, then we can always complete the embedding. Thus we shall refer to H as 3-*super*universal, and, more generally, we shall be concerned with the following and its variations.

Definition 1.1. For any positive integer k a simple graph G is k-*superuniversal* iff given any k-element simple graph H and any full embedding f of a subgraph of H into G, f can be extended to a full embedding of all of H into G.

(For applications of this notion to infinite graphs and metric spaces see [3].)

In what follows all graphs will be simple, and all embeddings will be full; i.e. graphs will have at most one edge between any two vertices, and if f is an embedding, then $f(a)$ will be adjacent to $f(b)$ if and *only* if a is adjacent to b.

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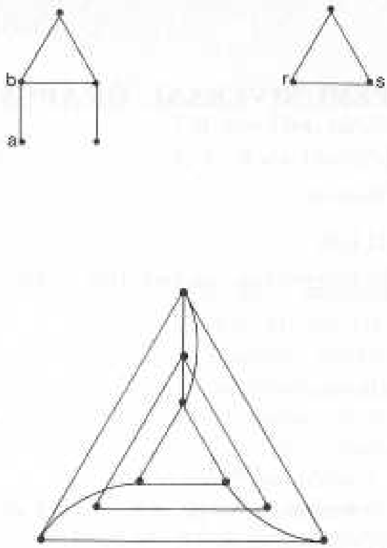

Fig. 1.

While technically a graph G should be thought of as an ordered pair $\langle V_G, E_G \rangle$ with V_G being the vertices and E_G the edges of G, we shall almost always use a single symbol, say G, to denote both the set of vertices of a graph and the graph itself. For any vertex v of any graph G we shall use v^+ to denote the set of vertices of G which are adjacent to v, and we shall use v^- to denote the set $G - v^+ - \{v\}$ unless G carries a coloring. In this latter case v^- will denote the set of vertices of G which are not adjacent to v and which are not of the same color as v. In either case, the dual of G will be the graph obtained by interchanging v^+ and v^- for each $v \in G$. For any set S we shall use $|S|$ to denote the number of elements in S.

In our calculations we shall frequently use logarithms both natural and to the base 2. To differentiate we let *log* denote logarithm to the base 2, and we always use *ln* to denote the natural logarithm.

The organization of this paper will be as follows. In Section 2 we shall consider k-superuniversal graphs as defined above and shall prove that for each k there exist arbitrarily large k-superuniversal graphs. We shall find upper and lower bounds, depending of course upon k, for the size of the smallest k-superuniversal graph, and we shall find various bounds on the possible valences of the vertices of these graphs. In Section 3 we shall consider graphs with colorings, and we shall present an appropriate version of k-superuniversality for these. We shall then consider the structure of these new graphs both with respect to the properties considered in Section 2 and with respect to properties of colorings. In Section 4 we shall collect various open problems. It should be noted that the methods we shall use in this paper can be applied to directed graphs, tournaments, etc., but the results do not seem to differ significantly from those obtained in the cases we shall consider here.

2. Superuniversal graphs

To check if a given graph is k-superuniversal is necessarily tedious, but the following criterion can frequently be used to simplify the computation, especially for k small. The proof is left to the reader.

Lemma 2.1. *A graph G is k-superuniversal iff it has at least k elements and given any $(k-1)$-element subset S of G and any subset T of S there is a vertex $v \in G - S$ for which $v^+ \cap S = T$.*

It is relatively easy to construct 3- and 4-superuniversal graphs; for example

Example 2.2. Let $\{S_i : i \leq n\}$ be any sequence of at least two sets each containing at least three elements. Then the graph G obtained by letting the vertices of G be the set $\prod S_i$ and defining two vertices to be adjacent iff they agree in at least one coordinate is easily seen to be 3-superuniversal.

Example 2.3. In the example above if there are at least three S_i containing at least four elements, and if two vertices are defined to be adjacent iff they agree on exactly one coordinate, then the resulting graph can be seen to be 4-superuniversal.

This method does not appear to generalize to 5-superuniversality. The problem comes up when we consider four sequences which differ only at only one coordinate. There is no way to define adjacency so as to allow a vertex which is adjacent to exactly two of these sequences, and, in fact, we know of no explicit examples of 5-superuniversal graphs.

On the other hand, in a strong sense, essentially all finite graphs are 5-superuniversal, and, as we see next, for any fixed k almost all finite graphs are k-superuniversal.

Theorem 2.4. *There exists a sequence $\{c_k : 2 < k < \infty\}$ such that $\lim(c_k) = 0$ and such that for any $n \geq (1+c_k)\ln(2)k^2 2^k$ there exists an n-element $(k+1)$-superuniversal graph. Furthermore, as $d \to \infty$ the ratio of $(1+d)\ln(2)k^2 2^k$-element graphs which are not $(k+1)$-superuniversal to all graphs of this size is of order essentially no greater than $2^{-k^2 d}$.*

Proof. For convenience we use $\exp_2(x)$ to abbreviate 2^x. Now let S be a set of n elements. The total number of possible edges over S is $\binom{n}{2}$, so the total number of possible graphs over S is $\exp_2[\binom{n}{2}]$.

We next compute an upper bound for the total number of non-$(k+1)$-superuniversal graphs over S. By Lemma 2.1 such a graph must contain a k-element set K and a subset L of K such that no vertex in $S-K$ is adjacent to every member of L and no member of $K-L$. Clearly, K may be chosen in $\binom{n}{k}$ ways and then L in 2^k ways. Once having chosen K we need not worry about possible edges whose endpoints are either both in K or both in $S-K$. There are $\binom{n}{2}$-$(n-k)k$ of these, thus yielding $\exp_2[\binom{n}{2}$-$(n-k)k]$ possibilities. Finally, since given K and L and any $v \in S-K$, all sets of edges from v into K are possible except for one, we have 2^k-1 possibilities for each such $v \in S-K$, and, therefore, $(2^k-1)^{(n-k)}$ altogether. Hence the total number of non-$(k+1)$-superuniversal graphs must be no greater than

$$\binom{n}{k} 2^k \exp_2\left[\binom{n}{2}-(n-k)k\right](2^k-1)^{(n-k)},$$

and the ratio of such graphs to all n-element graphs must be less than

$$\begin{aligned} R &= \binom{n}{k} 2^k 2^{-(n-k)k}(2^k-1)^{(n-k)} \\ &< n^k 2^k 2^{-(n-k)k} 2^{k(n-k)}(1-2^{-k})^{(n-k)} < n^k 2^k e^{-(n-k)2^{-k}}. \end{aligned}$$

We note that it is sufficient to prove that under our hypotheses R is less than 1, or, equivalently, that $\ln(R)$ is negative. But

$$\ln(R) < k\ln(n) + k\ln(2) - n2^{-k} + k2^{-k}.$$

Now put n in the form $(1+c_k)\ln(2)k^2 2^k$. Then we have

$$\begin{aligned} \ln(R) &< k\ln(1+c_k) + k\ln(\ln(2)) + 2k\ln(k) + k^2\ln(2) \\ &\quad + k\ln(2) + k2^{-k} - (1+c_k)k^2\ln(2) \\ &< -k^2(c_k\ln(2) - [\ln(1+c_k) + \ln(\ln(2)) + 2\ln(k) + \ln(2) + 2^{-k}]/k). \end{aligned}$$

Clearly, for suitable c_k this will be negative, and as $k \to \infty$ the required value of c_k will approach zero.

Furthermore, if in the form used above for n we substitute d for c_k where d is large compared to $\ln(k)/k$, then R will be of the order $2^{-k^2 d}$, and the proof is complete. $\square$

Thus for any k we have an upper bound for the size of the smallest k-superuniversal graph. To obtain lower bounds we begin with an observation which will be useful for other purposes as well and which is interesting in its own right. The proof follows immediately from the relevant definitions.

Theorem 2.5. *If G is k-superuniversal, then so is its dual, and for every vertex $v \in G$ both v^+ and v^- are $(k-1)$-superuniversal.*

Using this and considering the various possibilities, we find

Theorem 2.6.a. *Every 2-superuniversal graph has at least 4 vertices. There are 3 non-isomorphic 4-element 2-superuniversal graphs of which one is self-dual.*

Theorem 2.6.b. *Every 3-superuniversal graph has at least 9 vertices. There is a unique 9-element 3-superuniversal graph $K_3 \times K_3$ (where K_3 is the complete graph on 3 vertices), and it is, therefore, self-dual.*

Now we may combine Theorems 2.5 and 2.6.b to obtain a simple finite difference equation which, when we solve, yields

Theorem 2.7. *Every k-superuniversal graph has at least $10\,2^{k-3}-1$ vertices.*

Later we shall improve this result for $k \geqslant 10$.

We next consider questions concerning numbers of edges and valences of vertices of k-superuniversal graphs. We begin with lower bounds, and we treat the cases $k=2,3$ separately. The case $k=2$ is trivial; the valences need be no greater than 1 and the number of edges no greater than $|G|/2$. By Theorem 2.5 and 2.6.a every vertex of a 3-superuniversal graph must have valence at least 4, and we see now that this lower bound cannot be raised.

Theorem 2.8. *There exist arbitrarily large finite 3-superuniversal graphs G which have $3\,|G|-9$ edges and which have exactly 3 vertices with valence greater than 4.*

Proof. Let $V_n = \{v_i : 1 \leqslant i \leqslant 6n\}$ be a cycle; i.e. set v_i adjacent to v_{i+1}, and set v_n adjacent to v_1. Then add a set $U = \{u_1, u_2, u_3\}$ of three new vertices, and set u_i adjacent to v_j unless $i \equiv j$ (Mod 3). It is easily checked that the resulting graphs have all of the desired properties. □

Corollary 2.9. *There exist arbitrarily large infinite 3-superuniversal graphs which have exactly 3 vertices with valence greater than 4.*

Proof. To obtain an appropriate denumerable graph, replace the cycle V_n in the above construction by an infinite "line" $V_Z = \{v_i : -\infty < i < \infty\}$, and to obtain graphs of cardinality κ, simply add κ copies of V_Z. In both cases add U as above. □

In all of the above graphs there are three vertices with high valence. We now show that this number cannot be reduced.

Theorem 2.10. *Every 3-superuniversal graph with at least m^2-m+2 vertices contains at least three vertices with valence greater than m.*

Proof. Let G be a 3-superuniversal graph with at least m^2-m+2 vertices, and let v be any vertex of G with valence no greater than m. Now suppose that no member of v^+ has valence greater than m. Since v^+ must be 2-superuniversal, each vertex of v^+ must be adjacent to at least one other vertex of v^+. Hence there can be at most $m(m-2)$ vertices u of $G-v^+-\{v\}$ for which $u^+\cap v^+\neq\emptyset$. But 3-superuniversality implies that every vertex of $G-v^+-\{v\}$ has this property, and

$$|G-v^+-\{v\}|\geq m^2-m+2-m-1=m(m-2)+1.$$

Therefore, our assumption that each member of v^+ has valence no greater than m must be false.

Thus we have shown that every vertex of G either has valence greater than m or is adjacent to a vertex of valence greater than m. But if there are at most two vertices of valence greater than m, then every other vertex of G must be adjacent to at least one of these, and this violates 3-superuniversality. □

In particular, we have

Corollary 2.11. *Every 3-superuniversal graph with at least* 14 *vertices contains at least* 3 *vertices with valence greater than* 4.

The same argument applied to infinite graphs yields

Corollary 2.12. *Every infinite 3-superuniversal graph G contains at least* 3 *vertices with valence at least* $\mathrm{cf}(|G|)$. *In particular, every infinite 3-superuniversal graph contains at least* 3 *vertices with infinite valence.*

For k equal four and above the situation is quite different. No longer, even for fixed k, can the minimal valence of a k-superuniversal graph G be a fixed constant, but rather, it must be at least proportional to $\log(|G|)$. More precisely

Theorem 2.13. *If G is any k-superuniversal graph, $k\geq 4$, and v is any vertex of G, then the valence of v is at least* $2^{k-4}\log(|G|)$.

Proof. Let G and v be as above. Then let $V=\{v_i: 0\leq i\leq k-4\}$ be any set of distinct vertices of G such that $v_0=v$, and construct a sequence $\{S_i: 0\leq i\leq k-4\}$ of subsets of G inductively as follows. Let $S_0=v^+$, and let S_{i+1} be the smaller of the sets $S_i\cap v_{i+1}^+$ and $S_i\cap v_{i+1}^-$ choosing either if they are the same size. Also, let $S=S_{k-4}$. Clearly we have $|S|\leq|v^+|/2^{k-4}$. We note that since G is k-superuniversal, if we choose any two distinct vertices $u, w\notin V$, there will be a vertex in S which is adjacent to both u and w, and there will be another vertex of S which will be adjacent to u but not w. Thus the family $\mathcal{F}=\{u^+\cap S: u\notin V\}$ consists of distinct subsets of S no two of which are disjoint, and from this it follows easily that $|G|-(k-4)=|G-V|=|\mathscr{T}|<2^{|S|-1}$ which in turn implies

$$2^{|v^+|/2^{k-4}}>2\,|G|-2(k-4)>|G|. \quad \square$$

Essentially the same argument can be applied to discrete subgraphs of superuniversal graphs to obtain lower bounds on their complements.

Corollary 2.14. *If D is any discrete (or, by duality, complete) subgraph of a k-superuniversal graph G, then $|G-D|>2^{k-3}\log(|D|)$.*

We see next that in terms of order of magnitude these last two bounds are best possible. Our method of construction will be that of probabilistic graph theory which is described in [2].

Theorem 2.15. *For any k and arbitrarily large n there exist $(k+1)$-superuniversal n-element graphs which have fewer than $k2^k n\ln(n)$ edges and which contain a discrete subgraph whose complement has fewer than $k2^k \ln(n)$ vertices.*

Proof. For fixed k and large n let S be an n-element set, and let A be any subset of S. Construct the graph G^* by connecting each vertex in A to every other vertex in S. Now let G be the graph obtained by removing with probability $1/2$ each edge of G^*. A standard calculation similar to that used in the proof of Theorem 2.4 shows that if $|A|>k2^k\ln(n)$, then with probability greater than zero G will be $(k+1)$-superuniversal. It is clear that G has the requisite number of edges and that $S-A$ is the required discrete subgraph. $\square$

A surprising but immediate application of Theorem 2.13 is the improvement of Theorem 2.7 for $k\geqslant 10$. (The proof will be valid for $k\geqslant 4$, but the result is weaker than Theorem 2.7 for $k<10$.)

Theorem 2.16. *For $k\geqslant 10$, every k-superuniversal graph has at least $k2^{k-3}$ vertices.*

Proof. Assume $k\geqslant 10$, and let v be any vertex of a k-superuniversal graph G. By Theorem 2.13 we have $v^+\geqslant 2^{k-4}\log(|G|)$, and by duality we have the same lower bound on v^-. Combining these, we have $|G|>2^{k-3}\log(|G|)$. But by Theorem 2.7 we have $|G|>2^k$ which yields $\log(|G|)>k$. $\square$

We are note that if we apply this construction again using Theorem 2.16 instead of Theorem 2.7, we obtain the bound $|G|>2^{k-3}(k+\log(k)-3)$, but this is not significantly closer to the upper bound $k^2 2^k\ln(2)$ which we have from Theorem 2.4.

In another direction, we may look for lower gounds on the maximal valence appearing in a k-superuniversal graph. We find that in general we obtain two bounds depending upon the number of edges the graph contains.

Theorem 2.17. *If G is any n-element k-superuniversal graphk $k \geq 4$ with e edges and maximal valence m, then*

$$m/n > \max((n/2e)^{1/(k-2)}, (n \log(n)/2e)^{1/(k-4)}).$$

Proof. For any set $V \subseteq G$ define $V^+ = \{v : V \subseteq v^+\}$, define a vertex v to *cover* a set V iff $v \in V^+$, and define a pair $[v, V]$ to be a *j-covering pair* iff v covers V and $|V| = j$. We first note that by k-superuniversality every $(k-1)$-element subset of G must be covered, so there must exist at least

$$\binom{n}{k-1}$$

$(k-1)$-covering pairs in G. However, the total number of such pairs is

$$\sum_{v \in G} \binom{|v^+|}{k-1}$$

which is clearly less than $(2e/m)\binom{m}{k-1}$. (Where $2e/m$ is, of course, the total valence divided by the maximal valence.) Hence we have

$$\frac{2e}{m}\binom{m}{k-1} > \binom{n}{k-1}$$

which, because $m < n$, implies $(2e/m)m^{k-1} > n^{k-1}$, and thus gives us our first bound.

To obtain our second bound, we look at $(k-3)$-covering pairs. Suppose S is any $(k-3)$-element subset of G. If u and v are any two vertices in $G - S$, then by k-superuniversality there must be vertices r and s in S^+ such that r is adjacent to u and not v, while s is adjacent to v and not u. Since S is small compared to G, this clearly requires that $|S^+|$ be greater than $\log(n)$. Thus there must be at least

$$\binom{n}{k-3}\log(n)$$

$(k-3)$-covering pairs in G. But, as above, we also obtain an upper bound of $(2e/m)\binom{m}{k-3}$ such pairs, and this yields our remaining bound. $\square$

We note that we obtain only one bound for 4-superuniversal graphs and that for $k \geq 5$ the bound $(n/2e)^{1/(k-2)}$ is the sharper of the two bounds unless $2e < n(\log(n))^{(k-2)/2}$. We use this sharper bound to obtain a lower bound on m for all k-superuniversal graphs, $k \geq 3$.

Corollary 2.18. *Every k-superuniversal, $k \geq 3$, graph G has at least one vertex with valence greater than $|G|^{(k-2)/(k-3)}$.*

Proof. The case $k = 3$ follows from Theorem 2.10, so assume $k \geq 4$, and let G be any n-element k-superuniversal graph with e edges. Now suppose that $2e > n^{(2k-3)/(k-1)}$. Then since the maximal valence can be no less than the average valence, we must have $m > 2e/n$, and the result follows. But if $2e$ is no greater than $n^{(2k-3)/(k-1)}$, then the result follows immediately from the bound $m/n > (n/2e)^{1/(k-2)}$ obtained in Theorem 2.17. □

From this last proof, we see that if an n-element k-superuniversal graph is essentially homogeneous (its maximal valence is approximately equal to its average valence), then it must have at least $n^{(2k-3)/(k-1)}/2$ edges. We use probabilistic methods again to show that this Theorem 2.17, and Corollary 2.18 are at least close to best possible. The fact that almost all the graphs in question are homogeneous follows from [1].

Theorem 2.19. *For n large almost all n-element graphs with at most $n^{(2k-3)/(k-1)}((k-1)\ln(2n))^{1/(k-1)}/2$ edges are k-superuniversal, essentially homogeneous, and, therefore, have maximal valence approximately*

$$n^{(k-2)/(k-3)}((k-1)\ln(2n))^{1/(k-1)}.$$

On the other hand, we note that in the graphs constructed in Theorem 2.15 which were chosen because they had essentially the minimal number of edges possible, the maximal valence turns out to be of order $c\,|G|$. We see next that for $k \geqslant 5$ this is not due to the particular construction we used, but rather follows directly from the second bound in Theorem 2.17.

Corollary 2.20. *If $\mathcal{G}$ is any family of k-superuniversal graphs, $k \geqslant 5$, for which there exists a constant $c > 0$ such that each graph $G \in \mathcal{G}$ contains fewer than $c\,|G| \log(|G|)$ edges, then there exists a constant $d > 0$ such that each graph $G \in \mathcal{G}$ contains at least one vertex with valence greater than $d\,|G|$.*

We conclude this section with an observation on planarity. Suppose G is 4-superuniversal. By Theorem 2.5 and 2.6.b each of its vertices must have valence at least 9, and it is in fact even easy to see that it must contain a homomorphic copy of K_5 (the complete graph on 5 vertices) embedded in it. Thus it follows that

Theorem 2.21. *No 4-superuniversal graph is planar.*

The case of 3-superuniversal graphs is much more difficult. Purdy [5] has shown that no 3-superuniversal graph with more than 135 vertices can be planar, but the general question remains open. Purdy [6] has also shown that for every finite γ there is an upper bound on the number of vertices of a 3-superuniversal graph of genus γ.

3. Superuniversal chromagraphs

In this section we consider chromagraphs, that is, graphs with colorings. The problems will be similar to those considered in Section 2 except that embeddings will be required to preserve colorings. More formally

Definition 3.1. A chromagraph G is an ordered quadruple $\langle V_G, E_G, C_G, f_G \rangle$ such that $\langle V_G, E_G \rangle$ is a simple graph and f is a function from V_G onto C_G satisfying the condition $f(u) = g(u) \rightarrow \langle u, v \rangle \notin E_G$.

Intuitively, C_G is the set of colors used to color G, and f is the assignment of these colors to the vertices of G. The condition on f is simply the requirement that no two adjacent vertices be assigned the same color. As usual, we shall almost always use the same symbol to denote both G and V_G, and for any vertex $v \in V_G$ we let v^+ denote the set of vertices adjacent to v. However, as we stated earlier, we define v^- to be the set of vertices which are not adjacent to v and which have not been assigned the same color as v. The *dual* of G is again obtained by interchanging v^+ and v^- for every $v \in V_G$. Finally, for any $c \in C_G$ we define a set of the form $f_G^{-1}(c)$ to be a *color group*, and we define a chromagraph to be *balanced* iff each of its color groups contains the same number of vertices. We can now define superuniversality for chromagraphs.

Definition 3.2. A chromagraph G is *k-superuniversal* iff given any k-element chromagraph H for which $C_H \subseteq C_G$, and any color preserving embedding g of a subgraph of H into G, there is an extension of g which is a color preserving embedding of all of H into G.

As before, we have a somewhat more convenient criterion for determining k-superuniversality, and again we leave the proof to the reader.

Lemma 3.3. *A chromagraph G is k-superuniversal iff it has at least k color groups, and for every color group B of G, every $(k-1)$-element subset H of $G-B$, and every subset S of H, there is a vertex $v \in B$ for which $v^+ \cap H = S$.*

As with superuniversal graphs, for any fixed k essentially all balanced chromagraphs are k-superuniversal. The proof again uses the techniques of probabilistic graph theory, and the details are left to the reader. The precise theorem is

Theorem 3.4. *There exists a sequence $\{c_k : 2 < k < \infty\}$ such that $\lim(c_k) = 0$ and such that for any $n > (1 + c_k)\ln(2)k^2 2^k$ there exists a $(k+1)$-superuniversal chromagraph containing exactly $k+1$ n-element color groups. Furthermore, as n becomes larger than this bound, the probability that a chromagraph with $k+1$ n-element color groups is $(k+1)$-superuniversal rapidly approaches one.*

We next consider some basic properties of k-superuniversal chromagraphs and their consequences. Some will be analogues of those we have already found for ordinary superuniversal graphs, and, surprisingly, some will yield new information on the possible structure of these ordinary superuniversal graphs. The proofs are immediate and are left to the reader.

Theorem 3.5. *If G is any k-superuniversal chromagraph, then*

(a) *The dual of G is k-superuniversal.*

(b) *For every $v \in G$ both v^+ and v^- are $(k-1)$-superuniversal.*

(c) *Every color group of G contains at least 2^k vertices*

(d) *If the coloring of G is dropped, then G will remain a k-superuniversal graph even if edges are added between any number of pairs of vertices as long as both such vertices are from the same color group.*

(e) *If G contains exactly k color groups, then G does not contain a set of $k+1$ pairwise adjacent vertices.*

Combining (d) and (e) with Theorem 3.4, we have

Corollary 3.6. *For every k there exist*

(a) *Arbitrarily large k-superuniversal graphs which are not $(k+1)$-superuniversal.*

(b) *Arbitrarily large k-superuniversal graphs which can be partitioned into k equal discrete subgraphs.*

(c) *Arbitrarily large k-superuniversal graphs which can be partitioned into k equal complete subgraphs.*

We note that from (c) of Theorem 3.5 we have

Corollary 3.7. *Every k-superuniversal chromagraph has at least $k2^k$ vertices.*

As before, we shall use Theorem 3.5 and Corollary 3.7 to obtain stronger results for the case $k \geq 6$.

We note that from (d) of Theorem 3.5 it follows that all of our results on valences of k-superuniversal graphs can be applied directly to k-superuniversal chromagraphs. However, by applying the techniques of Section 2 directly to color groups we not only get somewhat stronger results overall, but we get more precise results in that they tell us, for example, about valences with respect to a given color group.

In what follows we shall always assume, for simplicity, that whenever we speak of a chromagraph as being k-superuniversal, it will have exactly k color groups. Similarly, whenever we speak of a k-superuniversal or $(k+1)$-superuniversal chromagraph, we shall always assume that it is at least 3-superuniversal, since 2-superuniversality is trivial. (A chromagraph containing exactly two color groups is 2-superuniversal iff it has neither an isolated vertex nor a vertex which is adjacent to every member of some color group.)

We begin with an analogue of Theorem 2.17.

Theorem 3.8. *If G is any $(k+1)$-superuniversal chromagraph, and B and R are any two color groups of G, then*

(a) *There is a vertex $v \in B$ for which $|v^+ \cap R| > |R|/|B|^{1/k}$.*

(b) *There is a vertex $v \in B$ for which $|v^+| > |G-B|/|B|^{1/k}$.*

Moreover, if G is balanced, then

(c) *There is a vertex $v \in B$ for which $|v^+ \cap R| > |B|^{(k-1)/k}$.*

(d) *There is a vertex $v \in B$ for which $|v^+| > (1+[\ln(k)-1]/k)\,|G|^{(k-1)/k}$.*

Proof. Let G be a $(k+1)$-superuniversal chromagraph, let R and B be color groups of G, let $r=|R|$ and $b=|B|$, let $B=\{v_i : i \leq b\}$, and let $m = \max(\{|v_i^+ \cap R| : i \leq b\})$. Then as in the proof of Theorem 2.17 we note that each k-element subset of R must be covered by a vertex from B, so we have

$$\binom{r}{k} \leq \sum_{i=1}^{b} \binom{|v_i^+ \cap R|}{k} \leq b\binom{m}{k}$$

which easily implies $r^k < bm^k$ or $m > r/b^{1/k}$. Similarly, if we set $g=|G|$ and $n = \max(\{|v_i^+| : i \leq b\})$, and we consider coverings of k-element subsets of $G-B$, we obtain the inequality $(g-b)^k < bn^k$ or $n > (g-b)/b^{1/k}$.

Now suppose that G is balanced. Then $r=b$, and our first result becomes $m > b^{(k-1)/k}$. We also have $g = b(k+1)$, and so our second result becomes

$$n > bk/b^{1/k} = kb^{(k-1)/k} = k[g/(k+1)]^{(k-1)/k} > k^{1/k}(1+1/k)^{(1-k)/k} g^{(k-1)/k}.$$

Finally, we note that

$$k^{1/k}(1+1/k)^{-(k-1)/k} > e^{\ln(k)/k} e^{-(1/k)(k-1)/k} > 1 + [\ln(k)-1]/k. \quad \square$$

For $k \geq 4$ we have lower bounds on the minimal possible valences of vertices of k-superuniversal chromagraphs. We begin with relative valences, i.e. valences with respect to given color groups.

Theorem 3.9. *If B is any color group of a k-superuniversal, $k \geq 4$, chromagraph G, and v is any vertex of $G-B$, then $|v^+ \cap B| > 2^{k-4}\log(|G-B|)$, and if $|B| \leq |G|/2$, then $|v^+ \cap B| > 2^{k-4}\log(|G|)$.*

Proof. Let $V = \{v_i : 0 \leq i \leq k-4\}$ be any sequence of vertices in $G-B$ such that $v_0 = v$, and, as in the proof of Theorem 2.13, choose a sequence $\{S_i : 0 \leq i \leq k-4\}$ of subsets of B as follows. Let $S_0 = v^+ \cap B$, let S_{i+1} be the smaller of the sets $v_{i+1}^+ \cap S_i$ and $v_{i+1}^- \cap S_i$, and let $S = S_{k-4}$. Now if u and w are any two vertices of $G-B-V$, then by k-superuniversality we have $u^+ \cap S \neq w^+ \cap S$, $u^+ \cap w^+ \cap S \neq \emptyset$, and $S \not\subseteq u^+ \cup w^+$. But this yields $|G-B-V| < 2^{|S|-1} - |S|$. Then since S is clearly larger than V, we have $|G-B|<2^{|S|-1}$, or $|G|+(|G|-2|B|)<2^{|S|}$, and since $|S|\leqslant|v^+\cap B|/2^{k-4}$, we are done. $\square$

From this we immediately obtain lower bounds on total valences.

Corollary 3.10. *If v is any vertex of a k-superuniversal, $k\geqslant 4$, chromagraph and B is the largest color group of G, then*

$$v^+>2^{k-4}\log(|G|^{k-2}|G-B|)>(k-2)2^{k-4}\log(|G|).$$

Furthermore, if no one color group of G is greater than $|G|/2$, or if $v\in B$, then $v^+>(k-1)2^{k-4}\log(|G|)$.

This corollary can now be used to show that there is a limit to the degree of imbalance of a superuniversal chromagraph. Note that the next two corollaries require only that k be no less than 3.

Corollary 3.11. *If B is any color group of a k-superuniversal, $k\geqslant 3$, chromagraph G, then $|B|>2^{k-3}\log(|G|)$.*

Proof. First suppose that $k\geqslant 4$. Let R be the smallest color group of G, and let v be any vertex of $G-R$. Then since $|R|$ is clearly less than $|G|/2$, we have $|v^+\cap R|>2^{k-4}\log(|G|)$. But by duality we also have $|v^-\cap R|>2^{k-4}\log(|G|)$, so $|R|>2^{k-3}\log(|G|)$. The case $k=3$ may be proven directly using the techniques involved in the proof of Theorem 3.9. $\square$

Finally, we may use these bounds to obtain lower bounds on the size of the smallest possible k-superuniversal chromagraphs. Although these bounds hold for all $k\geqslant 3$, they improve upon (c) of Corollary 3.6 only for $k\geqslant 6$.

Corollary 3.12. *If B is any color group of a k-superuniversal, $k\geqslant 3$, chromagraph G, then $|B|>(k+\log(k))2^{k-3}$, and hence $|G|>(k^2+k\log(k))2^{k-3}$.*

Proof. We already know from (c) of Theorem 3.5 that $|G|\geqslant k2^k$, so $\log(|G|)\geqslant k+\log(k)$. $\square$

As with ordinary superuniversal graphs, we may consider questions of planarity, but the situation here is somewhat simpler.

Theorem 3.13. *No 3-superuniversal chromagraph is planar.*

Proof. Let G be a 3-superuniversal chromagraph. It is sufficient to show that every vertex of G has valence at least 6. So suppose that there is a vertex $v\in G$ with valence less than 6, and let B be its color group. Then there must be a color group R such that $v^+ \cap R$ has exactly two elements, say r and s. (It cannot have fewer than two by (b) of Theorem 3.5.) But if w is any vertex in B, there must be some vertex in R which is adjacent to both w and v. Thus every vertex in B is adjacent to either r or s. But this violates the requirement of 3-superuniversality that there be a vertex in B which is adjacent to neither r nor s. $\square$

4. Open problems

Perhaps the most important remaining problem in this area is to find explicit constructions of superuniversal graphs and chromagraphs. We have mentioned examples of 3- and 4-superuniversal graphs, and it is not too difficult to construct 3-superuniversal chromagraphs, but beyond this we have no examples. A related problem whose solution might well lead to the construction of explicit examples is the problem of combining given superuniversal graphs into new superuniversal graphs. The same question can be asked with respect to chromagraphs, and we can further ask if there is a way of using ordinary superuniversal graphs to construct superuniversal chromagraphs.

Another major open question is that of for a given k determining for which n there exist n-element k-superuniversal graphs or chromagraphs. For example, our upper and lower bounds for the smallest such n differ by a factor of k. Furthermore, given the existence of an n-element k-superuniversal graph or chromagraph, we do not even know if there necessarily exists one with $n+1$ vertices. In particular, given a k-superuniversal graph or chromagraph, can one always add exactly one new vertex without destroying the k-superuniversality? It is possible to add one such vertex to the minimal 3-superuniversal graph and then to add one more, but we have no information about the general case.

There are two interesting open problems concerning 3-superuniversal graphs. One we have already mentioned, namely that of determining whether or not such a graph can be planar. The other is related to Theorem 2.8. This example, as well as others, leads us to believe that an n-element 3-superuniversal graph must have at least $3n-9$ edges. However, the best result to date in this direction is a very clever and involved proof by J. Pach that every such graph has at least $3n-30$ edges.

Finally, it would be of interest to know if any of our results such as 2.13 through 2.19 or 3.8 through 3.11 can be sharpened. In particular, does Corollary 2.20 hold for $k=4$?

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