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Ramsey-minimal graphs for multiple copies

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INTRODUCTION

Let F, G and H be finite, undirected graphs without loops or multiple edges. Write $F \to (G, H)$ to mean that if the edges of F are colored with two colors, say red and blue, then either the red subgraph of F contains a copy of G or the blue subgraph contains a copy of H. The class of all graphs F such that $F \to (G, H)$ will be denoted by $\mathscr{R}(G, H)$. A classical theorem of F. P. Ramsey guarantees that $\mathscr{R}(G, H)$ is non-empty.

The class $\mathscr{R}(G, H)$ has been studied extensively, particularly various minimal elements of the class. The *generalized Ramsey number* $r(G, H)$, which is the minimum number of vertices of a graph in $\mathscr{R}(G, H)$, has received the most attention. Surveys of recent results can be found in [1] and [7]. The *size Ramsey number* $\hat{r}(G, H)$, which is the minimum number of edges of a graph in $\mathscr{R}(G, H)$, was introduced in [4]. In the first section of this paper the size Ramsey number $\hat{r}(mK_{1,k}, nK_{1,l})$ will be calculated, where $sK_{1,t}$ denotes s disjoint copies of the star $K_{1,t}$. Moreover all graphs F with $\hat{r}(mK_{1,k}, nK_{1,l})$ edges for which $F \to (mK_{1,k}, nK_{1,l})$ will be determined. In the second section the following question will be considered. If $F \to (mG, nH)$, how many disjoint copies of G (or H) must F contain? In general, upper and lower bounds on the number of copies of G will be given, and in some special cases, exact results will be obtained.

Notation not specifically mentioned will follow that of Harary [6]. For a graph G, $V(G)$ is the vertex set and $E(G)$ is the edge set. The degree of a vertex v of G will be written $d_G(v)$. The maximum degree of a vertex of G will be denoted by $\Delta(G)$ and the minimum degree by $\delta(G)$. The notation for the independence number and the line independence number will be $\beta_0(G)$ and $\beta_1(G)$ respectively. The graph consisting of n disjoint copies of G will be written nG. The graph $G-v$ is the graph obtained from G by deleting a vertex v of G. Also as usual, [ ] is the greatest integer function and $|S|$ is the cardinality of the set S.

SIZE RAMSEY NUMBERS FOR STARS

For positive integers k and l, it is easily seen that $K_{1,k+l-1} \to (K_{1,k}, K_{1,l})$ and $K_3 \to (K_{1,2}, K_{1,2})$. It follows immediately that

$$(m+n-1)K_{1,k+l-1} \to (mK_{1,k}, nK_{1,l})$$

and

$$tK_3 \cup (m+n-t-1)K_{1,3} \to (mK_{1,2}, nK_{1,2})$$

for positive integers m and n and for $1 \leq t \leq m+n-1$. This implies

$$\hat{r}(mK_{1,k}, nK_{1,l}) \leq (m+n-1)(k+l-1),$$

which is one of two inequalities needed to prove the following theorem.

THEOREM 1: For positive integers k, l, m and n,

$$\hat{r}(mK_{1,k}, nK_{1,l}) = (m+n-1)(k+l-1).$$

Moreover if $G \to (mK_{1,k}, nK_{1,l})$ and has $(n+m-1)(k+l-1)$ edges, then $G=(m+n-1)K_{1,k+l-1}$ or $k=l=2$ and $G=tK_3 \cup (m+n-t-1)K_{1,3}$ for some $1 \leq t \leq m+n-1$.

If the theorem is not true, then for some k and l there exists a counterexample, and hence a minimal counterexample (no proper subgraph is a counterexample). Let $C_{k,l}$ denote the class of all such minimal counterexamples. If G is in $C_{k,l}$ then there exist positive integers m and n such that

1) $G \to (mK_{1,k}, nK_{1,l})$
2) $|E(G)| \leq (m+n-1)(k+l-1)$
3) $G \neq (m+n-1)K_{1,k+l-1}$ and $G \neq tK_3 \cup (m+n-t-1)K_{1,3}$ for $k=l=2$ and any t, $1 \leq t \leq m+n-1$.

The minimality of G implies that no proper subgraph H of G satisfies 1), 2) and 3) for any m and n. Of course any graph G in $C_{k,l}$ has parameters m and n associated with it. If such graphs are denoted by $C_{k,l}(m, n)$, then $C_{k,l}$ is the union of the classes $C_{k,l}(m, n)$.

To prove Theorem 1, it is sufficient to prove that $C_{k,l} = \phi$ for all k and l. The purpose of the next two lemmas is to describe properties of $C_{k,l}$ which will lead to showing it is empty. *For convenience it will be assumed throughout the remainder of this section that $k \geq l$.*

LEMMA 2: If $G \in C_{k,l}$, then
i) $|E(G)| > k+l-1$ and
ii) $\Delta(G) < k+l-2$.

PROOF: i) Assume $|E(G)| \leq k+l-1$. Then certainly $G \to (K_{1,k}, K_{1,l})$, $G \neq K_{1,k+l-1}$, and $G \neq K_3$ if $k=l=2$. If $|E(G)| \leq k+l-2$, the edges of G can be colored such that there exist no more than $k-1$ red edges and $l-1$ blue edges. If $E(G) = k+l-1$, then the coloring of any $l-1$ edges of G blue must leave the remaining edges forming a $K_{1,k}$. This cannot occur if G has two edges which are not incident. All pairs of edges of G being incident implies that $G = K_{1,k+l-1}$ or $G = K_3$. This contradiction completes the proof.

ii) Let G be a graph in $C_{k,l}(m, n)$ and assume v is a vertex of G of degree at least $k+l-1$. It will be shown that this leads to a contradiction. Either $m \geq 2$ or $n \geq 2$ by the first part of this lemma. The case $m \geq 2$ will be considered. A symmetric argument for $n \geq 2$ can be given.

If $G-v \not\to ((m-1)K_{1,k}, nK_{1,l})$, then the edges of $G-v$ can be colored such that there exists no red $(m-1)K_{1,k}$ and no blue $nK_{1,l}$. This coloring can be extended to G by coloring red the edges incident to v. In this coloring G contains no red $mK_{1,k}$ or blue $nK_{1,l}$, a contradiction. Therefore $G-v \to ((m-1)K_{1,k}, nK_{1,l})$.

The minimality of G implies that $G-v = (m+n-2)K_{1,k+l-1}$ or that $k=l=2$ and $G-v = tK_3 \cup (m+n-t-2)K_{1,3}$. Since

$$|E(G)| \leq (m+n-1)(k+l-1),$$

the vertex v has degree precisely $k+l-1$. This is of course true not for just a fixed vertex but for each vertex v of G of degree at least $k+l-1$. Using the fact that v is an arbitrary vertex of degree at least $k+l-1$, it is easily checked that this implies that $G = K_{2,k+l-1}$ or $k=l=2$ and $G = K_4$. Since $K_{2,k+l-1} \not\to (2K_{1,k}, K_{1,l})$ and $K_4 \not\to (2K_{1,2}, K_{1,2})$, this gives a contradiction.

The following lemma will be needed to describe some colorings of graphs used in the proof of Theorem 1.

LEMMA 3: If G is an element of $C_{k,l}(m, n)$, then there exists a sequence of vertices $v_1, v_2, \ldots, v_{n+m-1}$ of G such that $d_{G_{i-1}}(v_i) \geq k$ where $G_0 = G$ and $G_i = G - v_1 - v_2 \ldots - v_i$.

PROOF: Select v_1 to be a vertex of maximal degree in G and inductively select v_i to be a vertex of maximal degree in $G - v_1 - v_2 \ldots - v_{i-1} = G_{i-1}$. If the vertices $v_1, v_2, \ldots, v_{n+m-1}$ do not satisfy the conclusion of the lemma, then $\Delta(G_r) < k$ for some $r \leq n+m-2$. Assume such an r exists. Color the edges of G incident to v_i blue for each $i \leq n-1$. Color the remaining edges of G red. Clearly G contains no blue $nK_{1,l}$. Also G contains no red $mK_{1,k}$ since $\Delta(G_r) < k$ and every red $K_{1,k}$ must contain a vertex of the set

$\{v_m, \ldots, v_r\}$ (which might be empty). This contradiction completes the proof.

Let G be an element of $C_{k,l}(m, n)$. Two colorings of the edges of G will be described. Both colorings will be used to give lower bounds on the number of edges in G.

α-COLORING

Select arbitrary vertices $v_1, v_2, \ldots, v_{n-1}$ of G, and let r_i be the degree of v_i in $G - v_1 - \ldots - v_{i-1}$. Denote $G - v_1 - v_2 - \ldots - v_{n-1}$ by H. Color the edges incident to any v_i blue. Let $e_1 < e_2 < \ldots$ be an arbitrary ordering of the edges of H and color them sequentially using the following rule. An edge e_i is colored blue unless it is incident to a vertex that has $l-1$ edges of H incident to it that have already been colored blue. Then it is colored red.

In the α-coloring of G, every blue $K_{1,l}$ must contain one of the vertices $v_1, v_2, \ldots, v_{n-1}$. Thus G contains no blue $nK_{1,l}$. Therefore G, and hence H, must contain a red $mK_{1,k}$. Each edge of a red $K_{1,k}$ was colored red because one of its endvertices was incident to $l-1$ blue edges. Since $\Delta(H) \leq k+l-2$, the center of a red $K_{1,k}$ can be incident to no more than $l-2$ blue edges. Thus every vertex of a red $K_{1,k}$ except the center is incident to $l-1$ blue edges in H. Therefore the sum of the degrees in H of vertices of a red $K_{1,k}$ is at least $k+kl$. This implies that G has at least

$$\sum_{i=1}^{n-1} r_i + m(k+kl)/2 \text{ edges.}$$

β-COLORING

This coloring is the same as the α-coloring except the roles of red and blue, k and l, and m and n are interchanged. The β-coloring implies that G has at least

$$\sum_{i=1}^{m-1} r_i + n(l+lk)/2 \text{ edges.}$$

PROOF OF THEOREM 1: To prove the theorem it is sufficient to show that $C_{k,l} = \phi$ for all positive integers $k \geq l$. This will be done by an analysis of various cases of k and l. Let G be an element of $C_{k,l}(m, n)$ for some m and n.

$l=1$

Lemma 2 implies $\Delta(G) \geq k$. This contradiction proves that $C_{k,1} = \phi$.

$l \geq 4$, *or* $l=3$ *and* $k \geq 5$

Since G is in $C_{k,l}(m, n)$, $|E(G)| \leq (m+n-1)(k+l-1)$. The α-coloring in conjunction with Lemma 3 gives the following inequality

(a) $(n-1)k + m(k+kl)/2 \leq (m+n-1)(k+l-1)$.

Likewise the β-coloring and Lemma 3 imply

(b) $(m-1)k+n(l+lk)/2 \leq (m+n-1)(k+l-1)$.

These two inequalities can be rewritten in the following useful forms

(a′) $m(k-2) \leq 2(n-1)$

(b′) $n(l-2)(k-1) \leq 2(m-1)(l-1)$

It is straightforward to check that both inequalities (a′) and (b′) are never satisfied when $k \geq l \geq 4$ or when $l=3$ and $k \geq 5$. In fact (a′) implies $m<n$ while (b′) implies $m>n$. This contradiction completes the proof of this case.

$l=3,\ k=4$

Select vertices $v_1, v_2, \ldots, v_{m+n-1}$ as in Lemma 3. Lemma 3 guarantees that $d_{G_{i-1}}(v_i) \geq 4$ for all i, but in this case it can be assumed that $d_{G_{i-1}}(v_i) \geq 5$ for all i. To see this is true, assume $\Delta(G_r) \leq 4$ for some $r \leq n+m-2$. Color the edges red which are incident to $v_1, v_2, \ldots, v_t$ where $t = \max\{m-1, r\}$, and if $m \leq r$ color the remaining edges incident to $v_m, \ldots, v_r$ blue. The graph G_r can be embedded in a 4-regular graph H. By Petersen's Theorem [8], the graph H is 2-factorable with say factors H_1 and H_2. Color the edges of $H_1 \cap G_r$ red and the edges of $H_2 \cap G_r$ blue. The coloring just described implies $G \not\rightarrow (mK_{1,4}, nK_{1,3})$; this contradiction implies that $\Delta(G_r) \geq 5$.

In this case the α-coloring and the β-coloring give the following inequalities.

$$5(n-1)+8m \leq 6(m+n-1)$$
$$5(m-1)+15n/2 \leq 6(m+n-1).$$

Just as in the previous case, both inequalities cannot be satisfied simultaneously. This contradiction completes the proof of this case.

$l=k=3$

Lemma 2 implies that $\Delta(G) \leq 4$. By Petersen's Theorem [8] the graph G is the edge-disjoint union of two subgraphs each with no vertex of degree more than 2. Thus the edges of G can be colored such that no vertex is incident to more than two red edges or two blue edges. This implies $C_{3,3} = \phi$.

$l=2$

Lemma 2 implies $\Delta(G) \leq k$. It can be shown that $\delta(G) \geq 2$. To show this, suppose the contrary. Then there exists a vertex v of degree 1. Let w be the vertex of G adjacent to v in G. Thus w has degree at most $k-1$ in $G-v$. The minimality of G implies that the edges of $G-v$ can be colored such that there exists no red $mK_{1,k}$ and no blue $nK_{1,l}$. This coloring can be extended to G by coloring the edge vw. Since w has degree at most $k-1$ in $G-v$, the edge vw can be colored such that it is not in a red $K_{1,k}$ or a blue $K_{1,2}$. This implies $G \not\rightarrow (mK_{1,k}, nK_{1,2})$, a contradiction. Hence $\delta(G) \geqslant 2$.

Select vertices $v_1, v_2, \ldots, v_{m+n-1}$ as in the proof of Lemma 3. Each v_i is of degree at least k in G_i. Since $\Delta(G) \leqslant k$, the set $I = \{v_1, \ldots, v_{m+n-1}\}$ is an independent set of vertices each of degree k in G. Consider the bipartite graph B with parts I and $V(G) \setminus I$, where the edges of B are the edges of G between I and $V(G) \setminus I$. Each vertex of I has degree k in B. Since $\Delta(G) \leqslant k$, k is also an upper bound on the degree in B of vertices in $V(G) \setminus I$. Therefore a theorem of Philip Hall [5] implies that there exists a matching M of B using all of the vertices of I. For each i, $1 \leqslant i \leqslant m+n-1$, let w_i be the vertex matched with v_i. Let $W = \{w_1, w_2, \ldots, w_{m+n-1}\}$.

Select vertices $u_1, u_2, \ldots, u_t$ in $V(G) \setminus (I \cup W)$ such that the sum of their degrees is as large as possible and t is as large as possible but still no more than $n-1$. Color blue the edges of the matching M and all edges incident to any u_i. Color the remaining edges of G red. Since $t \leqslant n-1$, G does not contain a blue $nK_{1,2}$. Thus G contains a red $mK_{1,k}$. Let $u_n, u_{n+1}, \ldots, u_{m+n-1}$ be the centers of the m red graphs $K_{1,k}$. This set of centers is disjoint from I, W and $\{u_1, u_2, \ldots, u_t\}$, and each center has degree k in G. Hence $t = n-1$ and $d_G(u_i) = k$ for all i, $1 \leqslant i \leqslant m+n-1$. Let $U = \{u_1, u_2, \ldots, u_{m+n-1}\}$.

By assumption, $|E(G)| \leqslant (k+1)(m+n-1)$. Since $\delta(G) \geqslant 2$,

$$|E(G)| \geqslant (k(|I| + |U|) + 2|W|)/2 = (k+1)(m+n-1).$$

Therefore there must be equality: $V(G) = I \cup W \cup U$, $d_G(w) = 2$ for all w in W, and $d_G(z) = k$ for all z in $U \cup I$.

If $k \geqslant 3$, then a vertex v of I is adjacent to a vertex u of U. The vertex u could have been chosen in the matching M. This would imply that $d_G(u) = 2$, which contradicts the fact that $d_G(u) = 3$. Therefore $k = 2$ and G is a 2-regular graph with $3(m+n-1)$ vertices. If the edges of a cycle are colored red and blue alternately, the cycle will contain at most one monochromatic $K_{1,2}$. Since $G \rightarrow (mK_{1,2}, nK_{1,2})$, G must contain at least $m+n-1$ cycles. Hence $G = (m+n-1)K_3$, a contradiction to $G \in C_{2,2}$. This contradiction completes the proof of this case and of the theorem.

MULTIPLE COPIES

If $F \rightarrow (mG, nH)$, how many disjoint copies of G (or H) must F contain? Clearly F must contain at least m disjoint copies of G. If F is a complete graph then F contains $[|V(F)|/|V(G)|] \geqslant [r(mG, nH)/|V(G)|]$ disjoint copies of G. It is plausible that every F such that $F \rightarrow (mG, nH)$ contains at least $[r(mG, nH)/|V(G)|]$ disjoint copies of G. In some specific cases this will be shown to be true. A smaller general lower bound will be proved.

The magnitude of $r(mG, nH)$ is given by the following result which can be found in [2].

THEOREM 4: (Burr-Erdös-Spencer). If $|V(G)| = k$, $|V(H)| = l$, $\beta_0(G) = i$ and $\beta_0(H) = j$, then for some constant c,

$km+ln-\min(mi, nj)-1\leqslant r(mG, nH)\leqslant km+ln-\min(mi, nj)+c$,
where c depends only on G and H.

It will be established that if $F\to(mG, nH)$ and t is the left hand side of the inequality in Theorem 4, then either F contains at least t/k disjoint copies of G or at least t/l disjoint copies of H. In fact the following stronger statement will be proved.

THEOREM 5: If $F\to(mG, H)$, then $tG\subseteq F$ where

$$t=[(m|V(G)|+|V(H)|-\beta_0(H)-1)/|V(G)|].$$

PROOF: Assume to the contrary that $F\to(mG, H)$ but $tG\nsubseteq F$. Without loss of generality one can assume $(t-1)G\subseteq F$. Let $G_1, G_2, \ldots, G_{t-1}$ be a set of disjoint copies of G in F. Let S be the set of vertices contained in $G_m, \ldots, G_{t-1}$. It is possible that S is empty.

Color all of the edges of F incident with vertices of S blue and color all of the other edges red. In this coloring there exists no red mG and no blue H. There is no red mG, since this would be disjoint from the blue $(t-m)G$ and would imply $tG\subseteq F$. On the other hand, assume that there is a blue H. Such an H must have at least $V(H)-\beta_0(H)$ vertices in S since any collection of its vertices outside of S must be independent. Hence $|V(G)|(t-m)=|S|\geqslant|V(H)|-\beta_0(H)$. This inequality yields

$$t\geqslant(|V(H)|-\beta_0(H)+m|V(G)|)/|V(G)|.$$

Since t is an integer,

$$t\geqslant[(m|V(G)|+|V(H)|-\beta_0(H)+|V(G)|-1)/|V(G)|]=t+1,$$

a contradiction. This completes the proof.

There are several corollaries that follow immediately from this theorem.

COROLLARY 6: Let $|V(G)|=k$, $V(H)=l$, $\beta_0(G)=i$ and $\beta_0(H)=j$. If $F\to(mG, nH)$, then

(a) $sG\subseteq F$ where $s=[(mk+nl-nj-1)/k]$ and
(b) $tH\subseteq F$ where $t=[(mk+nl-mi-1)/l]$.

COROLLARY 7: If $|V(G)|=k$, $\beta_0(G)=i$ and if $m\geqslant n$, then $F\to(mG, nG)$ implies that F contains at least $[(mk+nk-ni-1)/k]$ copies of G.

Note that in the notation of Corollary 6, if

$$r(mG, nH)=km+ln-\min(mi, nj)-1$$

and $F\to(mG, nH)$, then either F contains a $[r(mG, nH)/k]G$ or a $[r(mG, nH)/l]H$ In [3] it was proved that $r(mK_2, nK_2)=2m+n-1$. These two facts give the following.

COROLLARY 8: If $F\to(mK_2, nK_2)$ and $m\geqslant n$, then the line independence number $\beta_1(F)\geqslant r(mK_2, nK_2)/2$ and this bound is the best possible.

The following is very similar to Corollary 8 but does require an additional argument in one case.

COROLLARY 9: If $F \to (mK_3, nK_3)$, then the number of independent triangles in F is at least $[r(mK_3, nK_3)/3]$ and this bound is the best possible.

PROOF: The complete graph on $r(mK_3, nK_3)$ vertices implies that the bound given is the best possible. One can show directly that if $F \to (K_3, K_3)$, then F must have at least two independent triangles. So assume $m \geq n$ and $m \geq 2$. In [2] it is shown that $r(mK_3, nK_3) = 3m + 2n$. Hence the corollary follows from Corollary 7 if $[(3m+2n-1)/3] = [(3m+2n)/3]$. Thus only the case when $[(3m+2n-1)/3] < [(3m+2n)/3]$, or equivalently, when n is a multiple of 3 remains to be considered.

Let $n = 3l$ and assume F has at most $[(3m+2n)/3] - 1 = m + 2l - 1$ independent triangles. It will be shown that this leads to a contradiction. Let $\{G_1, G_2, \ldots, G_{2l}\}$ be $2l$ disjoint triangles in F. Color the edges of each G_i blue as well as those edges with precisely one endvertex in a G_i, $1 \leq i \leq 2l$. Also color blue the edges between a G_i and a G_j if $1 \leq i, j \leq 2l-1$. Color the remaining edges red. In this coloring of F any blue triangle must contain at least two vertices from the vertices of the G_i, $1 \leq i \leq 2l$. Also the vertices of G_{2l} are contained in only one blue triangle, namely G_{2l}. Therefore there exists at most $[(6l-1)/2] = 3l - 1$ independent blue triangles. Any red triangle cannot use a vertex of any G_i, $1 \leq i \leq 2l$. Hence if F contains a red mK_3, there would exist $m + 2l$ independent triangles in F. This implies $F \not\to (mK_3, nK_3)$, a contradiction.

QUESTIONS

There are two questions left unanswered in this paper. The first involves Theorem 1 and whether this result can be extended to arbitrary star forests. This leads to the following conjecture:

If

$$F_1 = \bigcup_{i=1}^{s} K_{1,n_i} \text{ with } n_1 \geq n_2 \ldots \geq n_s$$

and

$$F_2 = \bigcup_{i=1}^{t} K_{1,m_i} \text{ with } m_1 \geq m_2 \ldots \geq m_t,$$

then $\hat{r}(F_1, F_2) = \sum_{k=2}^{s+t} l_k$ where $l_k = \max\{n_i + m_j - 1 : i + j = k\}$.

If $n_i = n$ for all i and $m_j = m$ for all j, then the conjectured value $\sum_{k=2}^{s+t} l_k$ agrees with the number $\hat{r}(sK_{1,n}, tK_{1,m})$ proved in section 1. The major question left open in section 2 of this paper is the following:

If $F \to (nG, nG)$, must F contain $[r(nG, nG)/|V(G)|]$ copies of G?

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