ON THE DISTRIBUTION OF VALUES OF ANGLES DETERMINED BY COPLANAR POINTS

J. H. CONWAY, H. T. CROFT, P. ERDŐS AND M. J. T. GUY

1.

Given a configuration $\mathscr{C}$ of n distinct points in the plane, no 3 collinear, then in all $N = \frac{1}{2}n(n-1)(n-2)$ angles are determined by triples of them. Define the functions

$$f(n, \alpha, \mathscr{C}), \quad g(n, \alpha, \mathscr{C})$$

to be the number of angles in $\mathscr{C}$ strictly greater than α, strictly less than α, respectively, where $0 < \alpha < \pi$. Further, attach superscripts $+$ to these functions (and the functions below) to denote the corresponding functions defined using inequalities in the wide sense. Next, put

$$f_1(n, \alpha) = \min_{\mathscr{C}} f(n, \alpha, \mathscr{C}); \quad g_1(n, \alpha) = \min_{\mathscr{C}} g(n, \alpha, \mathscr{C});$$

$$f_2(n, \alpha) = \max_{\mathscr{C}} f(n, \alpha, \mathscr{C}); \quad g_2(n, \alpha) = \max_{\mathscr{C}} g(n, \alpha, \mathscr{C});$$

(and similarly with superscripts). Finally,

$$F_i(\alpha) = \lim_{n \to \infty} f_i(n, \alpha)/N; \; G_i(\alpha) = \lim_{n \to \infty} g_i(n, \alpha)/N; \; (i = 1, 2).$$

We shall show below (Theorem 2) that these limits do indeed exist. There are some trivial logical relations between some of the functions F_i, G_i, F_i^+, G_i^+, thus:

$$G_2(\alpha) = 1 - F_1^+(\alpha); \; F_2^+(\alpha) = 1 - G_1(\alpha); \; F_2(\alpha) = 1 - G_1^+(\alpha); \; G_2^+(\alpha) = 1 - F_1(\alpha).$$

Further, we remark that the number of angles actually equal to any particular fixed $\alpha (0 < \alpha < \pi)$ is $o(n^3)$ as $n \to \infty$ (this is a weak consequence of Theorem 1 below, or else of a result of Croft [**2**], giving the exact upper bound); hence it follows that also

$$F_1^+ = F_1; \; G_1^+ = G_1.$$

We are thus reduced to consideration of the 2 functions F_1, G_1: from now on we usually drop embellishments and write F, G: crudely, $F(\alpha)$, $G(\alpha)$ denote " the ultimate proportion of angles necessarily $> \alpha$, $< \alpha$ " respectively.

2.

Exactly similar functions can be defined for configurations of points in 3 dimensions; we denote the functions corresponding to F, G by $\mathfrak{F}$, $\mathfrak{G}$; in general similar remarks to the 2-dimensional case are in order. We content ourselves with proving the following theorem, which obviates the need for the superscript $+$ in 3 dimensions, as in 2:

Received 30 November, 1977.

[J. LONDON MATH. SOC. (2), 19 (1979), 137–143]

THEOREM 1. *In 3 dimensions, for a configuration of n points, the number of angles exactly equal to any particular* α $(0 < \alpha < \pi)$ *is* $o(n^3)$, *as* $n \to \infty$.

Proof. Suppose not. Then by a combinatorial result of Erdős [**3**; Theorem 1], given any integer l, as large as we please, if n be sufficiently large $\left(n > n_0(l)\right)$, we may extract a subset $\{A_i, B_j, C_k;\, i, j, k = 1, 2, \ldots, l\}$ of $3l$ points, from our configuration of n points, with the property that

$$A_i \hat{B}_j C_k = \alpha \quad \text{(for each choice of } i, j, k).$$

Taking first $i = 1, j = 1$, we see that the points C all lie on a cone vertex B_1, axis $A_1 B_1$. Keeping $j = 1$, but letting $i = 2, 3, \ldots, l$ in turn, we obtain similar results; the upshot is that

either the points C lie on 2 straight lines (at most) both through B_1,

or the points A lie on one straight line through B_1. (*)

{The result holds true also in the special case $\alpha = \frac{1}{2}\pi$.} For $l \geqslant 5$, this result contradicts the non-collinearity condition. {Even if collinear points were admitted, use of (*), and the corresponding statements for $j = 2, 3, \ldots$, together with the conditions $\alpha > 0$, and the distinctness of the points, is soon seen to lead to a contradiction.}

3.

THEOREM 2. *The limits* $F_1(\alpha) = \lim_{n \to \infty} f_1(n, \alpha)$, $G_1(\alpha) = \lim_{n \to \infty} g_1(n, \alpha)$ *exist.*

Remark. The proof is of a quite general nature, and the idea can be applied to prove the existence of a limit in other problems.

Proof. We need consider only F. Let $\limsup f_1 = \Lambda$. Then, given an $\varepsilon > 0$, there exists an n_0 such that for *some* $n > n_0$,

$$|f_1(n, \alpha) - \Lambda| < \varepsilon.$$

Thus for this n (now fixed), there is a configuration $\mathscr{C}_n$ of n points with

$$|f(n, \alpha, \mathscr{C}_n) - \Lambda| < \varepsilon.$$

Take M " copies " of this configuration " on top of one another ", where M is very large; the distinctness (and non-collinearity) of points is preserved by having the copies of a typical point P—say $P_1, \ldots, P_M$—all distinct but very close to the original P; so sufficiently close that, if A, B, C are any 3 points of the original $\mathscr{C}_n$ with $A\hat{B}C > \alpha$, then $A_i \hat{B}_j C_k > \alpha$, where A_i, B_j, C_k are any copies of A, B, C, respectively. This gives us an admissible configuration $\mathscr{C}_{Mn}$ of Mn points. The number of angles greater than α in C_{Mn} is at least

$$(\Lambda - \varepsilon)\{Mn(Mn-1)(Mn-2) - \tfrac{1}{2}Mn(Mn-1)(M+M)\}, \tag{1}$$

the second term corresponding to angles such as $A_i \hat{B}_j B_k$ having at least 2 of their vertices copies of the same original. Indeed, for any ν with $Mn \leqslant \nu < (M+1)\,n$, we have a $\mathscr{C}_\nu$ with as many such angles as in (1), by merely taking the extra $\nu - Mn$ points arbitrarily. Then, on dividing by $\frac{1}{2}\nu(\nu-1)(\nu-2)$, we see that the proportion of angles greater than α in such a $\mathscr{C}_\nu$ is at least

$$\Lambda - \varepsilon - c/n - o_M(1),$$

for some (absolute) constant c, where $o_M(1)$ can be made arbitrarily small merely by making M large. Hence, if first n, and then M are chosen (suitably and) sufficiently large, both depending on ε, we obtain the result that

$$f(\nu, \alpha) > \Lambda - 2\varepsilon$$

for *all* sufficiently large ν. Since ε is arbitrary, the lower limit is equal to the upper limit Λ, as desired.

4.

We now pass to the functions F and G themselves. Easily we have:

(i) $F(\alpha)$ is a decreasing, and $G(\alpha)$ an increasing, function in $(0, \pi)$, both in the wide sense.

(ii) F is continuous on the left, G on the right. We prove merely the first, the second being similar. For a given α_0, and a given $\varepsilon > 0$, there exists an n, and a configuration $\mathscr{C}_n$ whose proportion of angles greater than or equal to α_0 is at most $F(\alpha_0) + \varepsilon$ (where we are using the results of Theorems 1 and 2). Now, for some sufficiently small δ (depending on ε, n and $\mathscr{C}_n$), all these angles are greater than $\alpha_0 - \delta$. Now, by a "copying" argument, as used in the proof of Theorem 2, we obtain *some* configuration $\mathscr{C}_{Mn}$, with arbitrarily large M, whose proportion of angles greater than $\alpha_0 - 2\delta$, say, is at most $F(\alpha_0) + \varepsilon + o_M(1)$, where $o_M(1)$ can be made small by sufficiently large choice of M. (We need $\alpha_0 - 2\delta$ rather than $\alpha_0 - \delta$ because of the small displacements caused by the "copying" operation.) Thus

$$F(\alpha_0 - 2\delta) \leqslant F(\alpha_0) + \varepsilon + o_M(1);$$

so, since ε is arbitrary, $\lim\limits_{\delta \to 0} F(\alpha_0 - 2\delta) \leqslant F(\alpha_0)$. Since F is decreasing, we have equality here, and F is continuous on the left.

(iii) $F(\alpha) = \frac{1}{3} \qquad (0 < \alpha \leqslant \frac{1}{3}\pi)$.

For, on the one hand, at least one angle of the 3 angles of each triangle determined by any triplet of any $\mathscr{C}$ is at least $\frac{1}{3}\pi$; and on the other hand a $\mathscr{C}_n$ given by taking n points all on a small circular arc of large radius gives $f_1(n, \alpha, \mathscr{C}_n) = \frac{1}{3}N$, for each n (and any fixed α).

(iv) $G(\alpha) = \frac{2}{3} \qquad (\alpha \geqslant \frac{1}{2}\pi)$.

For, on the one hand, 2 angles of the 3 of any triangle of any $\mathscr{C}$ are necessarily acute; and on the other hand, a $\mathscr{C}_n$ as in (iii) gives $g_1(n, \alpha, \mathscr{C}_n) = \frac{2}{3}N$, for each n (and any fixed α).

There are no other obvious values of F or G; we turn to estimating them, from above and below, for certain α of interest.

5. $F(\alpha)$, $\alpha = \frac{1}{2}\pi$. *How many angles are necessarily obtuse?*

Upper Bound.

THEOREM 3. $F(\frac{1}{2}\pi) \leqslant 4/27$.

Proof. We give a configuration with " few " obtuse angles. Take a triangle ABC, with angles $\hat{A} = \frac{1}{2}\pi - \varepsilon_1$, $\hat{B} = \varepsilon_2$, and with $BC = 1$ unit. Assuming as we may that n is divisible by 3, let $\mathscr{C}_n$ have: $\frac{1}{3}n$ points $A_i (i = 1, 2, \ldots, \frac{1}{3}n)$ close to A, $\frac{1}{3}n$ points B_i close to B, $\frac{1}{3}n$ points C_i close to C; each set arranged equally spaced on small circular arcs, thus:

the A_i on an arc through A of length ε_3, with centre B,

the B_i on an arc through B of length ε_4, with centre C,

the C_i on an arc through C of length ε_5, with centre A.

All the ε are small; the relationship between them will be chosen later. We consider triangles of various " types ": thus " type AAB " denotes generically triangles $A_i A_j B_k$. Altogether, there are $\frac{1}{3}N$ triangles. The following types are all acute-angled, provided the appropriate condition holds:

type ABC, provided $\varepsilon_3, \varepsilon_4, \varepsilon_5 \ll \varepsilon_1, \varepsilon_2$,

type AAB, provided $\varepsilon_4 \ll \varepsilon_3/n$,

type BBC, provided $(\varepsilon_1 + \varepsilon_2).\varepsilon_5 \ll \varepsilon_4/n$,

type CCA, provided $\varepsilon_3 . \varepsilon_1 \ll \varepsilon_5/n$.

We observe that these conditions are all compatible: e.g. take

$$\varepsilon_1 = \varepsilon_2 = n^{-5}, \varepsilon_3 = n^{-6}, \varepsilon_4 = \varepsilon_5 = n^{-8}.$$

Triangles of types ABB, BCC, CAA, of which there are $3N/81 + O(1)$ in number of each type and triangles of types AAA, BBB, CCC, of which there are $N/81 + O(1)$ in number of each type are all necessarily obtuse-angled. Since there is a (1–1)-correspondence between obtuse-angled triangles and the obtuse angles of $\mathscr{C}_n$, we have $(4/27)N + O(1)$ obtuse angles. Since n may be arbitrarily large, the result follows.

6.

Lower Bound.

THEOREM 4. $F(\frac{1}{2}\pi) \geqslant \frac{1}{9}$.

Proof. It is convenient to consider here $f^+(n, \frac{1}{2}\pi, \mathscr{C}_n)$, rather than f: take then such a $\mathscr{C}_n$ (for given n) that minimizes this, i.e. a configuration with the least number, which we may unambiguously denote by $f(n)$, of non-acute angles. Now remove a point from $\mathscr{C}_n$, leaving $n-1$ points, which determine at least $f(n-1)$ non-acute angles, by definition of f. Remove thus in turn each one of the n points of $\mathscr{C}_n$. Then the total number of non-acute angles counted in the n deleted configurations, each of $n-1$ points, is at least $n.f(n-1)$. But the total number that can be counted is: each of the $f(n)$ original angles, each counted just $n-3$ times (for $P_i \hat{P}_j P_k$ cannot be counted if and only if P_i or P_j or P_k is removed). So $(n-3)f(n) \geqslant nf(n-1)$; so, since $f(n)$ is an integer,

$$f(n) \geqslant \left\{\frac{n}{n-3} . f(n-1)\right\}. \tag{2}$$

where $\{x\}$ is the least integer not less than x. In addition to (2), we have the initial condition $f(4) = 1$ (every quadrilateral has a non-acute angle).

Define an integer-valued function $h(n)$ by

$$h(n) = \left\{\frac{n}{n-3}\,.h(n-1)\right\}, \qquad h(4) = 1. \tag{3}$$

Then, we show that

$$h(n) = \left\{\frac{n^2(n-3)}{18}\right\}$$

exactly. This follows by induction, for the truth of the hypothesis for $n-1$ implies

$$h(n) = \left\{\frac{n}{n-3}\left\{\frac{(n-1)^2(n-4)}{18}\right\}\right\} = \left\{\frac{n}{n-3}\left\{\frac{n(n-3)^2-4}{18}\right\}\right\}$$

$$= \left\{\frac{n^2(n-3)}{18} - \frac{2n}{9(n-3)}\right\} = \left\{\frac{n^2(n-3)}{18}\right\},$$

using the facts that $n^2(n-3)$ is congruent to 0, -2 or -4 mod 18, and that for $n \geqslant 6$, $2n/9(n-3) \leqslant 4/9$.

Thus $h(n) \sim \frac{1}{3}N$; but $f(n) \geqslant h(n)$, by induction. The result follows.

Remark (a). It is a nice fluke that (3) has an exact solution. It appears that most recurrence relations of the shape

$$k(n) = \left\{\frac{n}{n-a}\,.k(n-1)\right\}, \quad k(b) = c$$

for given integers a, b, c do not have a solution in closed form. An interesting problem is to characterize those which do.

Remark (b). If the $\{\ \}$ in (3) were dropped, and the resulting h used to determine a lower bound, we would indeed obtain one, but a much worse one, namely $F(\frac{1}{2}\pi) \geqslant \frac{1}{12}$.

Remark (c). If it were true that $f(5) = 4$, then we could drastically improve the estimate. However, this is false, though a counter-example is not very obvious; one may be constructed thus: distort a square $ABCD$ such that $\hat{A}$ becomes obtuse, $\hat{B}$, $\hat{C}$, $\hat{D}$ acute, and the diagonals cut (at a point O), not at right angles, but with $A\hat{O}B = \frac{1}{2}\pi + 3\varepsilon$, say ($\varepsilon > 0$); then construct a point E with $O\hat{A}E = O\hat{B}E$ ($= A\hat{E}B$) $= \frac{1}{2}\pi - \varepsilon$. Then the set $\{A, B, C, D, E\}$ contains but 3 non-acute angles.

7. *Other applications of the "lower bound" method.*

THEOREM 5. *$F(\frac{2}{3}\pi) \geqslant 5/171$. Also $\mathfrak{F}(\frac{1}{2}\pi) \geqslant 5/171$.*

Proof. For the first inequality we remark: any 6 coplanar points define some angle at least $\frac{2}{3}\pi$; for the result is obvious if the convex hull contains all 6, and if not, then an angle of at least $\frac{2}{3}\pi$ is defined at any internal point. The method of the previous section gives similar equations (2), (3) for new functions (that we may again call f, h) except that now the initial condition is $f(6) = h(6) = 1$; by calculation $h(20) = 100$, and so, for $n \geqslant 20$, by dropping brackets, we obtain

$$h(n) \geqslant \frac{n(n-1)(n-2).100}{20.19.18} = \frac{5N}{171}.$$

The first result follows. The second has identical working; for it happens to have the same initial condition, since any 6 points in 3 dimensions determine at least 1 non-acute angle: see Croft [**1**], or Grünbaum [**5**], or Schütte [**6**].

THEOREM 6. *For small* ε, $F(\pi-\varepsilon) \geqslant 2/8^{1/\varepsilon}$.

Proof. By a result of Szekeres [7], and Erdos and Szekeres [**4**], n points determine an angle greater than $\pi-\varepsilon$ if $\log_2 n > \varepsilon^{-1}$. So the initial condition to be fed into (3) for our new function h is $h(\{2^{1/\varepsilon}\}) = 1$. Dropping brackets, we obtain, for large n,

$$f(n) \geqslant h(n) \geqslant \frac{n(n-1)(n-2).1}{\{2^{1/\varepsilon}\}\{2^{1/\varepsilon}+1\}\{2^{1/\varepsilon}+2\}} \sim \frac{2N}{8^{1/\varepsilon}}.$$

The result follows.

We may also apply the "lower-bound" method to $G(\alpha)$.

THEOREM 7. (i) $G(\varepsilon) \geqslant 2\varepsilon^3/\pi^3$ *for small* ε,

(ii) $G(\frac{1}{3}\pi) \geqslant \frac{1}{3}$.

Proof. (i) By a result of Croft [**2**], n points determine some angle less than or equal to π/n (with equality only for the regular polygon); we feed in the initial condition $h(\{\pi/\varepsilon\}) = 1$ in (3), and working as in the last theorem, obtain $f(n) \geqslant 2N\varepsilon^3/\pi^3$.

(ii) 4 points determine at least 4 angles less than or equal to $\frac{1}{3}\pi$ (at least 1 in each triangle). The initial condition for our function h is $h(4) = 4$. By calculation $h(12) = 220$; so for larger n, dropping brackets,

$$h(n) \leqslant \frac{n(n-1)(n-2).220}{10.11.12} \sim \frac{1}{3}N.$$

The result follows.

8.

THEOREM 8. *F is discontinuous at* $\alpha = \frac{1}{3}\pi$; *in particular,* $F(\frac{1}{3}\pi+\varepsilon) \leqslant 7/27$.

Proof. To prove the second clause of the theorem it suffices, given an arbitrary ε, to produce a configuration of n points with n arbitrarily large, the proportions of angles greater than $\frac{1}{3}\pi+\varepsilon$ in which is at most 7/27. Without essential loss of generality, we may, as before, take n divisible by 3; now place the n points on a circle, with one-third of them near each of the vertices of an inscribed equilateral triangle ABC. With obvious notation the points are A_i, B_i, C_i $(1 \leqslant i \leqslant \frac{1}{3}n)$. We speak of triangles of types AAB, and so on, as before. If the points in each collection are close enough to A, B, and C (a constant multiple of ε will be found sufficient), then all angles of triangles of type ABC are not greater than $\frac{1}{3}\pi+\varepsilon$. However, every other triangle is obtuse-angled with acute angles less than $\frac{1}{3}\pi+\varepsilon$; and the proportion of these triangles is $\frac{7}{9}$; the result follows. The first clause of the theorem follows on recalling that $F(\frac{1}{3}\pi) = \frac{1}{3}$.

9.

Many problems of interest on the functions F and G remain; in particular:

(i) are they discontinuous for any other α, or for infinitely many values of α?

(ii) are they *strictly* monotonic (outside the trivial intervals noted in §4)?—or maybe even—strongly in the opposite sense—neither possesses at any point a non-zero derivative?

(iii) What are the true orders of magnitude of $F(\pi-\varepsilon)$ and $G(\varepsilon)$, for small ε?

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J. H. Conway,
Department of Pure Mathematics,
16 Mill Lane,
Cambridge.

H. T. Croft,
Peterhouse,
Cambridge.

P. Erdős,
Hungarian Academy of Sciences,
Budapest.

M. J. T. Guy,
Computing Laboratory,
Corn Exchange Street,
Cambridge.