ANOTHER PROPERTY OF 239 AND SOME RELATED QUESTIONS

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Introduction.

There are many questions that we can ask about the expression of a factorial as the product of k factors:

$$n! = a_1 a_2 \ldots a_k \tag{0}$$

We might assume that the factors lie in the interval $[n+1, 2n]$ and that they are either distinct or not:

$$n < a_1 < a_2 < \ldots < a_k \le 2n \tag{1}$$

or

$$n < a_1 \le a_2 \le \ldots \le a_k \le 2n \tag{2}$$

On the other hand, we might require that the a_i be distinct, but remove the upper bound and perhaps relax the lower bound as well:

$$n < a_1 < a_2 < \ldots < a_k \tag{3}$$

or

$$1 < a_1 < a_2 < \ldots < a_k \tag{4}$$

or we might only require that the a_i be positive integers:

$$a_1 \le a_2 \le \ldots \le a_k \tag{5}$$

In a previous note [3] it was proved that (1) has only a finite number of solutions. Here we enumerate all solutions and prove

Theorem 1. There are no solutions of (0) and (1) for $n > 239$.

We also outline a proof of

Theorem 2. Solutions for (0) and (2) can be found for all $n > 13$.

Finally we make assumption (3) and denote the minimum value of a_k by $f(n)$, i.e. $f(n)$ is the smallest integer for which $n!$ can be represented as the product of distinct integers greater than n, the largest of which is $f(n)$. We then prove

Theorem 3. There are constants $0 < c_1 < c_2$ such that

$$2n + \frac{c_1 n}{\ln n} < f(n) < 2n + \frac{c_2 n}{\ln n}$$

for all sufficiently large n.

No doubt there is a constant c such that

$$f(n) = 2n + \frac{cn}{\ln n} + o(\frac{n}{\ln n})$$

and perhaps this can be shown by a more careful application of our method.

Some other questions. The problem of determining min $(a_k - a_1)$ is also of interest. Assume $k > 1$ (else $a_k = n!$); then it seems likely that $a_k - a_1 > cn$ under condition (4) or (5), i.e. whether we assume the a_i to be distinct or not. At present such a theorem seems far beyond our means. The real difficulty occurs when k is small; in particular when $k = 2$. It has never been proved that

$$n! = a_1(a_1+1)$$

has no solutions for $n > 3$. In fact

$$n! = u^\alpha(u+1)^\beta$$

seems to have no solution larger than $4! = 2^3 3$. A long outstanding conjecture is that

$$n! = (x-1)(x+1)$$

has no solution for $n > 7$.

We determine $\min\,(a_k - a_1)$ for small values of n under each of the conditions (4) and (5), i.e. with and without the assumption that the a_i are distinct. Perhaps the general answers, under assumptions (4), (2) and (5) are respectively

$$¿\quad \min(a_k - a_1) = n + o(n) \quad ?$$

$$¿\quad \min(a_k - a_1) = \tfrac{2}{3}n + o(n) \quad ?$$

$$¿\quad \min(a_k - a_1) = \tfrac{1}{2}n + o(n) \quad ?$$

Under condition (3) with $k > 1$ we believe that, for sufficiently large n,

$$¿\quad a_k - a_1 > n \quad ?$$

If we assume that $a_1 \le n$, then it is easy to see that

$$\min(a_k - a_1) > n - C \ln n \tag{6}$$

by looking at the highest power of two which divides $n!$ If $2^\alpha \| n!$ then $\alpha > n - (\ln n)/(\ln 2)$. On the other hand if $2\beta \| a_k!/(a_1 - 1)!$ then $\alpha < \beta < a_k - a_1 + c \ln a_k$ and (6) follows immediately. Moreover (6) is not far from being best possible, since if $n = s! - 1$, then

$$n! = \frac{(n+1)!}{s!} = \prod_{i=1}^{n-s+1} (s+i)$$

so that

$$a_k - a_1 < n - \frac{\ln n}{\ln \ln n}$$

Is it true, under condition (4) with $k > 1$, that

$$\min(a_k - a_1) = n - 2 \tag{7}$$

for infinitely many values of n? It would be nice to decide this elementary question. For $4 < n < 16$, $\min(a_k - a_1) < n - 2$, while for $n = 16$ the equality (7) holds. In fact it semms certain that when $n = 2^{\nu}$ is a large enough power of two, then (7) holds for the following reason. Unless one of the a_i is a multiple of $2^{\nu+1}$ we must have $a_k - a_1 \geq n - 2$. if one of the a_i is a multiple of $2^{\nu+1}$ we must have $a_1 > n$. Now if $a_1 < n^{1+\varepsilon}$ we can prove that $a_k - a_1 > n + cn/\ln n$ and although we cannot yet handle the case $a_1 > n^{1+\varepsilon}$ it is very likely that it gives smaller values of $a_k - a_1$.

Suppose that the a_i are distinct, that $k > 1$ and that $a_1 a_2 \ldots a_k / n!$ is an integer with no prime factors greater than n. Is it true that

$$\min_i (a_k - a_1) < n - 2 \quad ?$$

Perhaps this can be proved, since an old and simple result says that $(2n)!/n!(n+3)!$ is an integer for almost all n.

If we only assume (5) then clearly every prime $p \leq n$ must have a multiple pq such that $a_1 \leq pq \leq a_k$. This condition is not sufficient, but we can prove that it does suffice provided $a_k < Cn$ and n is sufficiently large, $n > n_0(C)$. Because the condition $a_k < Cn$ can no doubt be very much weakened (we don't know by how much) we do not give the lengthy proof.

We examined a problem which we find quite interesting. Let $p_1 < p_2 < \ldots < p_l$ be a set of l primes. Denote by $A(p_1, \ldots, p_l)$ the smallest integer such that every interval of length A contains l *distinct* integers $a_i \equiv 0 \pmod{p_i}$, $1 \leq i \leq l$. It seemed to us that for every C there is a set of $l = l(C)$ primes with $A(p_1, \ldots, p_l) > Cp_l$. This problem can be specialized in the following ways.

Let $h_1(m,n)$ be the smallest integer for which every prime $p \le n$ has a multiple among the numbers $m+i$, $1 \le i \le h_1$, i.e. h_1 is the least integer for which

$$\prod_{p \le n} p \text{ divides } \prod_{i=1}^{h_1} (m+i)$$

And let $h_2(m,n)$ be the smallest integer for which every prime power $p^\alpha \le n$ has a multiple among the $m+i$, $1 \le i \le h_2$. Finally, let $h_3(m,n)$ be the smallest integer such that

$$n! \quad \text{divides} \quad \prod_{i=1}^{h_3} (m+i)$$

Then it is easy to see that $h_1(m,n) \le h_2(m,n) \le h_3(m,n)$. Put

$$H_j(m,n) = \min_{1 \le u \le m} h_j(u,n), \quad j = 1,2,3.$$

For fixed n, as m increases, each of the $H_j(m,n)$ decreases (from near n) to 1. We will investigate these functions in a later paper, if we live. Here is a typical problem.

Let t_n be the shortest interval $< n(1+\varepsilon)$ which contains a multiple of each prime $\le n$. (This definition is deliberately vague to allow for possible irregularities in the distribution of primes). Determine or estimate the smallest m for which $H_j(m,n) < t_n$. We can show that tnis m is greater than n^{1+c} and that if one assumes conjectures about the distribution of primes that are probably true but hopeless to prove, then $m > n^2/(\ln n)^c$.

Let $1 = u_1 < u_2 < \ldots$ be the sequence of integers all of whose prime factors are $\le n$, let u_r be the smallest u_i greater than m and let l be the smallest integer for which every prime $\le n$ divides $\prod_{i=0}^{l} u_{r+i}$. We conjecture that the equation

$$(8) \qquad n! = \prod_{j=0}^{l} u_{r+j}^{\alpha_j} \quad , \ \alpha_j \geq 0$$

is usually solvable, but if we insist that each α_j is 0 or 1, then (8) is not usually solvable. Note that for small values of m, $h_1(m,n) = u_{r+l} - m$, i.e. for each prime $p \leq n$ there is an $i \leq h\ (m,n)$ with $n + i \equiv 0 (\mathrm{mod}\, p)$. Determine the least $m = m(n)$ for which $h_1(m,n) < u_{r+l} - m$. E.g, if $n = 10$, to see that $m(10) = 30$ we note that $h_1(30,10) = 5$ (every prime less than ten divides one of 31,32,33, 34,35) but $u_{r+l} = 36$ (not 35) since 33 has a prime factor 11 and so is not a u_i and it is easy to check that $h_1(m,10) = u_{r+l} - m$ for $m < 30$. It should be possible to prove that $m(n)$ is of order about n^2.

For most values of m, the values of $h_j(m,n)$ are not much smaller than n since usually there is a prime very close to n which has a multiple which is very little smaller than m. In fact, as $x \to \infty$,

$$\frac{1}{x} \sum_{m=1}^{x} h_j(m,n) \longrightarrow \alpha_j(n) \quad , \quad j = 1,2,3$$

and it is not hard to prove that $\alpha_j(n)/n \to 1$ as $n \to \infty$. Can $\alpha_j(n)$ be determined explicitly?

To conclude this collection of problems we formulate a few related questions and conjectures. Write

$$B(n,k) = \prod_{i=1}^{k} (n+i).$$

It seems certain that for $k > 1$, $l > 1$, $m \geq n + k$, the equation $B(n,k) = B(m,l)$ has only a finite number of solutions (in fact very few). Unfortunately, even special cases of this conjecture are usually quite intractable.

A well known theorem [2,8] of Pillai-Szekeres-Brauer states that if $1 \leq l \leq 16$ then l consecutive integers always include one which is relatively prime to the others and this is false for every

$l > 16$. For $l = 17$ the integers 2184, 2185, ..., 2200 form the simplest counterexample. In a previous paper [5] we found an example of an interval $[a,b]$ where a and b are relatively prime and every $a+i$ $0 \le i \le b-a$, has a common factor with the product ab. We do not know for which values of $b-a$ this is possible. We also asked the following question which is probably very difficult. Is it true that for every r there are k_r consecutive integers $n+1, n+2, \ldots, n+k_r$ so that to each i, $1 \le i \le k_r$, there corresponds a $j \ne i$, $1 \le j \le k_r$ for which the g.c.d. $(n+i, n+j)$ has at least r distinct prime factors.

Finally an old problem of P. Erdős. Take $k = n$ in (0) and (5) and determine or estimate $\max a_1$. It was conjectured that

$$\max a_1 > \frac{n}{e}(1-\varepsilon) \quad ?$$

for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$. Selfridge and Straus believe that they can prove that $\max a_1 > n/3$ for $n > n_0$. It is easy to see that

$$\max a_1 < \frac{n}{e} - \frac{cn}{\ln n}$$

Erdős, Selfridge and Straus recently proved that

$$\max a_1 = \frac{n}{e} + o(n).$$

Proof of Theorem 1. We consider the identity

$$\binom{2n}{n} n! = (n+1)(n+2) \ldots (2n) \tag{9}$$

and notice that the problem of expressing $n!$ as the product of distinct factors in the interval $[n+1, 2n]$ is exactly complementary to that of expressing $\binom{2n}{n}$ in a similar way. Now $\binom{2n}{n}$ contains all the primes in this interval, so we will concern ourselves only with those which are less than n (and hence less than $2n/3$). For example

$$\binom{28}{14} = (23\times 19\times 17) \times 5^2 \times 3^3 \times 2^3$$

and the product $5^2 \times 3^3 \times 2^3$ can be arranged as $15 \times 18 \times 20$, the product of three numbers in the interval. So

$$14! = 16 \times 21 \times 22 \times 24 \times 25 \times 26 \times 27 \times 28.$$

There are two common circumstances in which the method shows that we are doomed to failure. For example, if $n = 20$,

$$\binom{40}{20} = (37\times 31\times 29\times 23) \times 13 \times 11 \times 7 \times 5 \times 3^2 \times 2^2.$$

The primes between $2n/3$ and $n/2$ (here 13 and 11) have to be paired with 2 or 3. If we form the *smallest* possible products, 13×2, 11×2 and then 7×3, we are left with 5×3 which is too small. So if there is a solution, this part of the calculation contains less than four factors. But if we form the *largest* possible products, 13×3, 11×3 and 7×5, we are still left with 2^2, so all attempts produce a number of factors strictly between 3 and 4. We denote this situation by the symbol 3+.

On the other hand, look at the case $n = 81$.

$$\binom{162}{81} = (157.151...83)53.47.43.41.31.29.23.17.11.7^2.5.2^3$$

Here we have to pair the primes 53,47,43 and 41 with a 2 or a 3 and there are only three such factors available. We denote this situation by writing $4 > 3$. More generally, even where there are sufficient factors 2 and 3, we may run out of the next batch of small factors. If $n = 121$ we have

$$\binom{242}{121} = (241.239...127)79.73.71.67.61.47.43.41.31.13.5^2.2^5$$

Here the five primes 79,...,61 need a multiplier 2 or 3, while 47,43,41 need a multiplier 3,4 or 5 and 31 needs a multiplier 4,5,6 or 7. There are enough twos for the first five, but only two factors 5 with which to accommodate the next three and 31. We write this $9 > 7$ (i.e. $5+3+1 > 5+2$).

Table 1 shows the values of n, $1 \leq n \leq 242$, for which there are no solutions, together with one of these two reasons. For $n \geq 243$ there is always a shortage of small factors.

1		34	3+	75	7+	108	9>6	137	7>6	162	6>4	184	9>7	211	11>9
2		36	3>2	79	7+	109	10>8	138	10+	163	6>5	185	14+	212	11>10
4	0+	37	3+	80	7+	110	8>7	139	10+	164	7>5	186	9>8	213	16+
5	1+	38	3+	81	4>3	111	10>8	140	10>9	165	7>5	190	13+	214	16+
7	1+	41	4+	82	7+	112	5>3	141	10+	166	6>5	192	7>3	216	9>8
9	1+	42	3+	83	8+	113	10>9	142	9+	167	12>10	193	7>4	217	12>10
10	0+	45	4+	84	7+	114	10>8	143	9+	168	6>5	195	9>8	220	14+
12	1+	46	4+	85	7+	115	9>8	144	9>8	169	9>8	196	6>5	225	9>7
13	2+	49	5+	87	7+	118	10+	147	8>7	170	10>8	197	10>9	226	8>7
16	1+	50	5+	88	7+	119	10+	148	8>7	171	7>6	198	7>5	227	12>11
17	2+	53	5+	91	7+	120	5>4	150	11+	172	7>5	199	7>6	228	9>8
19	3+	54	4+	92	7+	121	9>7	151	12+	173	10>9	200	7>5	231	15+
20	3+	57	3+	93	7+	128	7>6	152	10>9	174	7>6	201	7>5	232	12>11
23	3+	58	3+	96	7+	129	9+	153	11>9	175	10>7	202	6>5	234	14>13
24	2+	62	6+	97	7+	130	6>5	154	11>8	176	10>7	204	12>11	235	15+
26	3+	65	6+	100	6+	132	9>8	155	11>9	177	9>7	205	15+	236	15+
27	2+	66	5+	101	7+	133	10>9	156	9>7	178	9>8	206	15+	237	15+
30	3+	70	5+	102	7+	134	10+	157	9>8	180	9>6	208	10>8	240	15>14
31	3+	71	6+	105	9+	135	7>6	160	7>6	181	9>7	209	11>10	241	17+
33	4+	72	6+	106	8+	136	5>4	161	10>8	182	9>8	210	11>8	242	17+

Table 1. Values of n for which there are no solutions, and why.

Table 2 gives the complementary set of values of n for which there *are* solutions, together with the numbers of solutions. There are no solutions if $n > 329$. For $n = 239$ there is a record number of 92967 solutions, accounting for more than three-quarters of the total of 119126 solutions.

3	1	25	2	47	11	63	7	78	1	104	36	127	10	187	11	219	648
6	1	28	1	48	10	64	2	86	18	107	6	131	165	188	1983	221	6
8	1	29	2	51	4	67	1	89	64	116	10	145	12	189	6	222	313
11	1	32	2	52	4	68	35	90	4	117	2	146	42	191	6	223	13855
14	1	35	1	55	1	69	5	94	11	122	237	149	302	194	20	224	360
15	1	39	2	56	3	73	12	95	103	123	28	158	32	203	3255	229	54
18	3	40	1	59	2	74	2	98	6	124	1	159	338	207	9	230	288
21	1	43	3	60	8	76	6	99	16	125	97	179	120	215	696	233	1419
22	1	44	17	61	1	77	108	103	8	126	30	183	3	218	882	238	392
																239	92967

Table 2. Values of n for which there are solutions, and numbers of solutions.

Proof of Theorem 2. We start from the same identity (9) and multiply each odd primepower factor of $\binom{2n}{n}$ by the appropriate power of two to bring it into the interval $[n+1, 2n]$. These products will all be distinct and we may cancel them with the corresponding members of $(n+1)(n+2)\ldots(2n)$. It remains to deal with the extra power of two, say $2^m = 2^{kq+r}$ where $n+1 \le 2^k \le 2n$ and $|r| \le k/2$. This may be regarded as q factors 2^k which can serve as q of the a_i (since condition (2) no longer requires them to be distinct) and 2^r remaining to be disposed of. For large enough n it is always possible to dispose of r twos by multiplying some of the $[n+1, 2n]$ by suitable factors. For example, if $n = 20$,

$$\binom{40}{20} \times (20)! = 21 \times 22 \times 23 \times \ldots \times 39 \times 40$$

$$(37.31.29.23.13.11.7.5.3^2.2^2)(20)! = 21.22.23.\ldots.39.40$$

$$(37.31.29.23.26.22.28.40.36)(20)! = (21.22.23.\ldots.39.40)2^7$$

$$(20)! = 21.24.25.27.30.32.33.34.35.38.39.2^7$$

Write 2^7 as 32×2^2 and absorb the 2^2 by multiplying 21 by 4/3, 24 by 3/2, 25 by 8/5 and 32 by 5/4 giving

$$(20)! = 28.36.40.27.30.40.33.34.35.38.39.32$$

Of course, there is at least one repetition, 40, since we know there is no solution for $n = 20$ under condition (1).

To be sure of finding solutions for large enough n we will restrict ourselves to multipliers 3/2 and 4/3 if twos need to be inserted, or to 2/3 and 3/4 if r is negative and twos need to be deleted. We illustrate with the example $n = 110$:

$$(10) \quad \binom{220}{110} = (211.197...113)73.71.67.61.59.43.41.37.31.29.23.19.13.11.7.5.3.2^5$$

so we cancel the primes between 110 and 220 from both sides of the equation

$$(11) \qquad \binom{220}{110} 110! = 111.112...220$$

and multiply the remaining odd prime(power)s, 73,71,...,3, in (10) by the appropriate powers of two to bring them into the interval [111,220]:

(12) 146,142,134,122,118,172,164,148,124,116,184,152,208,176,112,160,192.

Then we delete these numbers from the right of equation (11). This uses 1+1+1+1+1+2+2+2+2+2+3+3+4+4+4+5+6 = 44 twos and these, apart from the five twos in (10), must be replaced. Write 2^{44-5} as $(2^7)^5 2^4$ or $(2^7)^6 2^{-3}$. In the first case we include five factors 128 and insert the other four twos by multiplying 111,117,123 and 129 by 4/3 (i.e. replacing them by 148,156,164 and 172) and 114,120,126 and 132 by 3/2 (replacing them by 171,180,189 and 198). In the second case we include six factors 128 and delete the excess of three twos by multiplying 207,201 and 195 by 2/3 (becoming 138,134 and 130) and 204,180 and168 by 3/4 (becoming 153,135 and 126). Note that 192 occurs in the list (12) which has been deleted, and is not available for multiplication by 3/4.

The first case multiplies odd multiples of three by 4/3 and multiples of six by 3/2. These must be chosen from the interval $[n+1, 4n/3]$ and $\lfloor n/18 \rfloor$ of each type of number is available with the possible exception of just one multiple of six which may have been deleted. The second case multiplies odd multiples of three by 2/3 and multiples of twelve by 3/4. These must be chosen from the interval $[4(n+1)/3, 2n]$ and $\lfloor n/18 \rfloor$ multiples of twelve are available, again with a possible exception (192 in the example) which may have been deleted when disposing of the power of three from $\binom{2n}{n}$. Notice that we can alternatively absorb the multiplier 3/4 in a number which is four times a prime in the interval $[2n/5, n/2]$ because such primes do not occur in $\binom{2n}{n}$. In the example, 188 and 212 could have served in place of two of 204, 180 and 168.

In any case, n will certainly be large enough if $\lfloor n/18 \rfloor - 1 \geq |r|$ where we chose $|r| \leq \lfloor k/2 \rfloor$ and $k = \lfloor \mathrm{lb}(2n) \rfloor$ where "lb" is the binary (base 2) logarithm. There are enough numbers to absorb the multipliers if $n \geq 72$ and smaller values of n can easily be checked. We need consider only those entries which occur in Table 1.

4! 1+ (2 factors $\geq 5^2$ are too big, 1 factor ≤ 8 is too small)

5! 2+ (6^3 too big, 10^2 too small)

7! 3+ (10.14 must occur, then 8^2 is too big, 14 is too small)

$9! = 10.12^2.14.18$, or, more compactly, $12^3.14.15$

10! 5+ (14 must occur, then $12^2 15^2 16$ too big, $18^2 20^2$ too small)

$12! = 14.15^2.16.18.22.24 = 14.15.16.18^2.20.22 = 15^2.16^2.18.21.22$

13! 7+ (22.26 must occur, then $14.15^2.16.18x$ is too big if $x > 12$,
while $21.24^2 25y$ is too small if $y < 36$)

For $n = 1, 2, 4, 5, 7, 10$ and 13 there are no solutions. There are solutions for the entries not in Table 1: $3! = 6$, $6! = 8.9.10$, $8! = 12.14.15.16$, $11! = 12.18.20^2.21.22 = 14.18^2.20^2.22 = 15.16.18.20.21.22$; for $n = 9$ and 12 given above, and it is easy to construct solutions for $n > 13$ up to where the method described earlier takes over.

Proof of Theorem 3. Recall that $f(n) = \min a_k$ subject to (0) and (3). We first establish the lower bound

$$2n + \frac{c_1 n}{\ln n} < f(n). \tag{13}$$

Let the standard form for $n!$ be $\prod p^{\alpha p}$ where the product is over all primes not exceeding n. For the primes between $n/2$ and $2n/3$ the exponent $\alpha_p = 1$, because $2p > n$. so $2p$ and $3p$ cannot both be among the a_i. Suppose that β_2 multiples of 2 and β_3 multiples of 3 do not occur as a_i, i.e. these are missing from the product

$$\prod_{i=1}^{f(n)-n} (n+i) \tag{14}$$

Then, by the prime number theorem,

$$\beta_2 + \beta_3 = n(1+o(1))/6 \ln n \tag{15}$$

the number of primes p, $n/2 < p < 2n/3$. Let γ_2, γ_3 be the exponents of 2 and 3 occurring in the product (14) so that

$$\gamma_2 - \beta_2 \le \alpha_2 \qquad \text{and} \qquad \gamma_3 - \beta_3 \le \alpha_3, \tag{16}$$

the exponents of 2 and 3 in $n!$ It is well known that

$$\begin{aligned} \alpha_2 &= n + O(\ln n), & \alpha_3 &= \tfrac{1}{2}n + O(\ln n) \\ \gamma_2 &= f(n) - n + O(\ln n), & \gamma_3 &= \tfrac{1}{2}(f(n)-n) + O(\ln n) \end{aligned} \tag{17}$$

and (15), (16) and (17) yield (13) with c_1 arbitrarily close to 1/9.

To obtain the upper bound

$$2n + \frac{c_2 n}{\ln n} > f(n) \tag{18}$$

we return to the identity (9) and note that $\binom{2n}{n} = \prod p^{\alpha}$, where the product is taken over some of the primepowers less than $2n$. The primepowers between n and $2n$ may be cancelled from (9) and the primepowers less than n can be multiplied by appropriate powers of two, as in the proof of Theorem 2, and also cancelled from (9) leaving an identity

$$n! = 2^m \prod (n+i)$$

where the product runs over most of the values of i from 1 to n. The power of 2 is absorbed by doubling the first m values of $n+i$, so that $f(n) < 2n + 2m(1+o(1))$ and it remains to estimate m.

Write $z = n(1+o(1))/\ln n$, so that the prime number theorem asserts that z is the number of primes less than n. There are no primes p, $2n/3 < p < 2n$, which divide $\binom{2n}{n}$. The number of prime divisors of $\binom{2n}{n}$ between $n/2$ and $2n/3$ is $z/6$. There are none between $2n/5$ and $n/2$, and generally none between $2n/(2\omega+1)$ and n/ω, while the number between $n/(\omega+1)$ and $2n/(2\omega+1)$ is $z/(\omega+1)(2\omega+1)$. The power of 2 needed to bring such primes into the interval $[n+1,2n]$ is 2^y where $\omega+1 \le 2^y < 2\omega+1$, or $y = \lfloor \mathrm{lb}(2\omega+1) \rfloor$ and the total number of twos required is at most

$$\sum_{\omega=1}^{\infty} \lfloor \mathrm{lb}(2\omega+1) \rfloor z/(\omega+1)(2\omega+1).$$

That is

$$m \le \left[\left(\frac{1}{2.3}\right) + \left(\frac{2}{3.5} + \frac{2}{4.7}\right) + \left(\frac{3}{5.9} + \frac{3}{6.11} + \frac{3}{7.13} + \frac{3}{8.15}\right) + \left(\frac{4}{9.17} + \ldots\right.\right] z$$

and (18) follows for sufficiently large n with $c_2 = 1.7$, since the series in the bracket has sum less than 0.85.

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