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PROJECTIVE $(2n, n, \lambda, 1)$-DESIGNS

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Abstract: A projective $(2n, n, \lambda, 1)$-design is a set $\mathscr{D}$ of n element subsets (called blocks) of a $2n$-element set V having the properties that each element of V is a member of λ blocks and every two blocks have a non-empty intersection. This paper establishes existence and non-existence results for various projective $(2n, n, \lambda, 1)$-designs and their subdesigns.

AMS (MOS) Subject Classifications: Primary 05B05; Secondary 05B30, 05C65.

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This paper deals exclusively with *projective* (v, k, λ, t)*-designs* for $k = n$, $v = 2n$ and $t = 1$. A projective (v, k, λ, t)-design is a set $\mathscr{D}$ of k-element subsets (called blocks) of a v-element set V having the properties that (i) each t-element subset of V is a subset of λ blocks in $\mathscr{D}$ and (ii) every two blocks in $\mathscr{D}$ have a non-empty intersection. We omit the word projective from this point on. A (v, k, λ, t)-design is *primitive* if it contains no proper (v, k, λ', t)-design. A $(2n, n, \lambda, 1)$-design is often referred to as a λ*-tuple cover,* for example, a $(2n, n, 2, 1)$-design is called a double cover.

Although this paper is essentially self-contained, it is an out-growth and extends some of the work of Ehrenfeucht, Faber and Simmons (1979) and the reader is referred to that paper for a discussion of the origins of this work and related references. Most of the results follow strongly from the Erdös–Ko–Rado theorem (1961).

The main results in this paper are as follows.

Theorem 5. *A* $(2n, n, \lambda, 1)$*-design exists if and only if* $n \geq 3$ *and either*
(a) n *is a power of* 2 *and* $2 \leq \lambda \leq [\frac{1}{4}\binom{2n}{n}] - 1$, *or*
(b) n *is not a power of* 2 *and* $2 \leq \lambda \leq \frac{1}{4}\binom{2n}{n} - 2$ *or* $\lambda = \frac{1}{4}\binom{2n}{n}$.

Theorems 11 and 12. *Every* $(2n, n, \frac{1}{4}\binom{2n}{n}, 1)$*-design contains a triple cover and at least*

$$\left[\frac{1}{2(2n-1)} \frac{1}{4}\binom{2n}{n}\right]$$

disjoint double covers.

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Theorem 13. *Every $(2n, n, \lambda, 1)$-design $\mathscr{D}$ with*

$$\lambda > \left(1 - \frac{1}{2n-1}\right)\frac{1}{4}\binom{2n}{n} + 2(n-1)$$

contains a $(2n, n, \gamma, 1)$-design for some γ with $2 \le \gamma \le 2n$, and hence is not primitive.

Theorem 14. *Every $(2n, n, \lambda, 1)$-design with $\lambda \le \frac{1}{2}\frac{1}{4}\binom{2n}{n} - 2$ can be extended to a $(2n, n, \lambda+2, 1)$-design.*

We proceed with the results and proofs.

Theorem 1. *If $n = 2^l$ with $l \ge 1$, there is a $(2n, n, [\frac{1}{4}\binom{2n}{n}] - 1, 1)$-design $\mathscr{D}_n$ which is the disjoint union of $(2n, n, 2, 1)$-designs.*

Proof. We form the design $\mathscr{D}_n$ on $V = \{1, 2, \ldots, 2n\}$ as follows. Let $S_k = \{2k-1, 2k\}$, $k = 1, 2, \ldots, n$. For each k, we partition $V - S_k = X \dot{\cup} Y$ into two equal parts of size $n-1$ in such a way that for each $l < k$, either $S_l \subseteq X$ or $S_l \subseteq Y$. The number of such partitions is

$$N_k = \frac{1}{2}\sum_{r=0}^{k-1}\binom{k-1}{r}\binom{2n-2k}{n-2r-1}.$$

(We can construct a given partition by first choosing r of the S_l's with $l < k$ to be contained in X $(0 \le r \le k-1)$. This can be done in $\binom{k-1}{r}$ ways. The remaining S_l's with $l < k$ are made elements of Y. We must now specify $n-2r-1$ of the remaining $2n-2k$ elements of V to be included in X. This can be done in $\binom{2n-2k}{n-2r-1}$ ways. Since the roles of X and Y can be reversed, each partition is counted twice.) Now, N_k is always even since $\binom{2n-2k}{n-2r-1}$ is always even and in the case where there are an odd number of summands (k odd), the term

$$\binom{k-1}{\frac{1}{2}(k-1)}\binom{2n-2k}{n-1}$$

is divisible by 4. Consider two of the partitions $X \dot{\cup} Y = V - S_k = U \dot{\cup} Z$. A double cover is formed by the blocks

$$\{2k-1\} \cup X, \quad \{2k-1\} \cup Y, \quad \{2k\} \cup U, \quad \{2k\} \cup Z.$$

The union of all these double covers and their complements contains all the n element subsets of V except those which are the union of $\frac{1}{2}n$ of the n sets S_k. By induction, inside the remaining $\binom{n}{n/2}$ sets, a $(n, \frac{1}{2}n, [\frac{1}{4}\binom{n}{n/2}] - 1, 1)$-design can be found which is the disjoint union of double covers, unless $n = 4$, in which case the desired $(4, 2, 0, 1)$-design is empty. The design $\mathscr{D}_n$ is then taken to be the union of all of these double covers.

Theorem 2. *If $n = 2^l$ with $l \ge 1$, then $(2n, n, \lambda, 1)$-designs exist if $2 \le \lambda \le [\frac{1}{4}\binom{2n}{n}] - 1$.*

Proof. By Theorem 1, the theorem holds if λ is even. Now we claim that if $l\geq 2$, then the design $\mathscr{D}_n$ in Theorem 1 can be decomposed into two triple covers and double covers. As in the proof of Theorem 1, this fact will be true by induction if it can be verified for the $(8,4,16,1)$-design $\mathscr{D}_4$. Consider the three double covers in the decomposition of the $(8,4,16,1)$-design $\mathscr{D}_4$ formed by the blocks

$$\begin{array}{llll} A_1=\{1,3,4,5\}, & A_2=\{1,6,7,8\}, & A_3=\{2,3,4,6\}, & A_4=\{2,5,7,8\}, \\ B_1=\{2,5,6,8\}, & B_2=\{2,3,4,7\}, & B_3=\{1,3,5,7\}, & B_4=\{1,4,6,8\}, \\ C_1=\{3,5,6,8\}, & C_2=\{1,2,3,7\}, & C_3=\{4,5,7,8\}, & C_4=\{1,2,4,6\}. \end{array}$$

The blocks A_1, A_2, A_4, B_2, C_1, C_4 form one triple cover and the blocks A_3, B_1, B_3, B_4, C_2, C_3 form another. This proves the claim. By combining one of these triple covers with the proper number of double covers, $(2n,n,\lambda,1)$-designs can be constructed for all odd λ except $[\frac{1}{4}\binom{2n}{n}]-2$. We show that a $(2n,n,\lambda,1)$-design with $\lambda=[\frac{1}{4}\binom{2n}{n}]-2$ can be constructed from $\mathscr{D}_n$ by replacing four of the blocks in one of the constructed triple covers by two blocks not in $\mathscr{D}$. As above, it is necessary only to consider the $(8,4,16,1)$-design $\mathscr{D}_4$. We replace the blocks A_1, B_2, C_1, C_4 in the first triple cover by the blocks $\{1,2,3,4\}$ and $\{3,4,5,6\}$ to form the double cover A_2, A_4, $\{1,2,3,4\}$, $\{3,4,5,6\}$. This proves the theorem.

Theorem 3. *Suppose $n=2^lq$ with q an odd integer greater than 1. There is a $(2n,n,\frac{1}{4}\binom{2n}{n},1)$-design $\mathscr{D}_n$ which is the union of $(2n,n,2,1)$-designs, unless $q=2^m+1$, in which case there is a $(2n,n,\frac{1}{4}\binom{2n}{n},1)$-design which is the union of $(2n,n,2,1)$-designs and one $(2n,n,q,1)$-design.*

Proof. As in the proof of Theorem 1, we let $S_k=\{2k-1,2k\}$ and for each k, we form N_k partitions of $V-S_k$ into two equal parts X and Y so that either $S_i\subseteq X$ or $S_i\subseteq Y$, where

$$N_k=\frac{1}{2}\sum_r\binom{k-1}{r}\binom{2n-2k}{n-2r-1}.$$

There are two cases to consider: n odd and n even. First we consider the case n even. In this case, N_k is even for all k by the same argument as before. Also, as before, a collection of disjoint double covers can be formed having the property that the union of these double covers and their complements contains all n element subsets of V except those which are the union of $\frac{1}{2}n$ of the n sets S_k. By induction, since $n\neq 2^l$, we see that this case reduces to the case where n is odd. In the case where n is odd, we analyze the situation as follows. Since

$$\binom{2n-2k}{n-1-2r}=\binom{2n-2k}{n-2k+2r+1},$$

we have

$$\binom{2n-2k}{n-1-2r}+\binom{2n-2k}{n+1+2r-2k}\equiv\binom{n-k}{\frac{1}{2}(n-1)-r}+\binom{n-k}{\frac{1}{2}(n+1)-k+r}\pmod 4.$$

If $k=2r+1$,

$$\binom{k-1}{\frac{1}{2}(k-1)}\binom{2n-2k}{n-k}\equiv 0\equiv\binom{k-1}{\frac{1}{2}(k-1)}\binom{n-k}{\frac{1}{2}(n-k)}\pmod 4.$$

Thus

$$2N_k\equiv\sum_r\binom{k-1}{r}\binom{n-k}{\frac{1}{2}(n-1)-r}\equiv\binom{n-1}{\frac{1}{2}(n-1)}\pmod 4.$$

If $n=2^m+1$, all of the N_k are odd, otherwise all of the N_k are even. In case all of the N_k are even, as above, a collection of disjoint double covers can be formed having the property that the union of these double covers and their complements contains all n element subsets of V except those which are the union of $\frac{1}{2}n$ of the n sets S_k. In this case, n is odd, so the exception is vacuous. The union of these double covers is the design $\mathscr{D}_n$. If $n=2^m+1$, a $(2n,n,n,1)$-design is formed using one of the N_k partitions for each k as follows. Let

$$T_k=S_{(n+2k-1)/2}\cup S_{(n+2k+1)/2}\cup\cdots\cup S_n\cup S_1\cup\cdots\cup S_{k-1}.$$

The design is formed by the $2n$ blocks $\{2k-1\}\cup T_k$, $\{2k\}\cup T_k$, $k=1,2,\ldots,n$. The remainder of the N_k-1 partitions are apportioned into double covers in the same manner as the other cases. Since n is odd, the union of all these double covers and the n-tuple cover form a $(2n,n,\frac{1}{4}\binom{2n}{n},1)$-design $\mathscr{D}_n$. This completes the proof of this theorem.

Theorem 4. *If n is not a power of 2, then $(2n,n,\lambda,1)$-designs exist if $2\le\lambda\le\frac{1}{4}\binom{2n}{n}-2$ or if $\lambda=\frac{1}{4}\binom{2n}{n}$.*

Proof. The theorem is true for $n=3$ by inspection. If $n\ge 5$, three of the double covers constructed in the decomposition of $\mathscr{D}_n$ in the proof of Theorem 3 can be rearranged to form two triple covers as in the proof of Theorem 2. By forming the right combination of double covers and triple covers (and the 2^m+1-tuple cover in the case where $n=2^l(2^m+1)$), a λ-tuple cover can be constructed for all λ such that $2\le\lambda\le\frac{1}{4}\binom{2n}{n}-2$. Details are left to the reader.

Theorem 5. *A $(2n,n,\lambda,1)$-design exists if and only if $n\ge 3$ and either*

(a) *n is a power of 2 and $2\le\lambda\le[\frac{1}{4}\binom{2n}{n}]-1$, or*

(b) *n is not a power of 2 and $2\le\lambda\le\frac{1}{4}\binom{2n}{n}-2$ or $\lambda=\frac{1}{4}\binom{2n}{n}$.*

Proof. Only the necessity of these conditions remains to be proved. To prove this theorem, it seems easier to prove a slightly more general result (Theorem 6) which first requires a definition.

Definition. A $(2n,n,\lambda,1)$-*predesign* is a set $\mathscr{D}$ of n-element subsets of a $2n$-element set V having the properties

(i) $(1/2n)\sum_{v\in V}\lambda(v,\mathscr{D})=\lambda$, where $\lambda(v,\mathscr{D})$ is the number of sets in $\mathscr{D}$ containing v.

(ii) $\lambda(v, \mathscr{D}) \equiv \lambda(w, \mathscr{D}) \pmod 2$ for all v and w in V,

(iii) every two members of $\mathscr{D}$ have a non-empty intersection.

Remark 1. Note that if $\mathscr{D}$ is obtained from a predesign by replacing some blocks by their complements, $\mathscr{D}$ is a predesign.

Theorem 6. *A $(2n, n, \lambda, 1)$-design exists if and only if a $(2n, n, \lambda, 1)$-predesign exists. A $(2n, n, \lambda, 1)$-predesign exists if and only if $n \geq 3$ and either*

(a) *n is a power of 2 and $2 \leq \lambda \leq [\frac{1}{4}\binom{2n}{n}] - 1$, or*

(b) *n is not a power of 2 and $2 \leq \lambda \leq \frac{1}{4}\binom{2n}{n} - 2$ or $\lambda = \frac{1}{4}\binom{2n}{n}$.*

Proof. We have already shown the existence part of this theorem in Theorems 2 and 4. It remains to demonstrate the non-existence part. Let $\alpha = \alpha_n = \frac{1}{4}\binom{2n}{n}$.

Case I. The non-existence of a $(2n, n, [\alpha], 1)$-predesign when n is a power of 2.

Suppose a $(2n, n, [\alpha], 1)$-predesign $\mathscr{D}$ exists. Since $[\alpha] = \alpha - \frac{1}{2}$, the number of blocks either in $\mathscr{D}$ or complementary to a block in $\mathscr{D}$ is $4\alpha - 2 = \binom{2n}{n} - 2$. We may assume that the two remaining n-element subsets of V have the form $\{1, 2, \ldots, n\}$ and $\{n+1, n+2, \ldots, 2n\}$. By replacing blocks by their complements if necessary, we may assume that every block in $\mathscr{D}$ contains the element 1. Thus the blocks of $\mathscr{D}$ are exactly the sets of the form $\{1\} \cup S$ with $S \subseteq V - \{1\}$, $|S| = n-1$ and $S \neq \{2, 3, \ldots, n\}$. Hence

$$\lambda(1, \mathscr{D}) = \binom{2n-1}{n-1} - 1,$$

$$\lambda(j, \mathscr{D}) = \binom{2n-2}{n-2} - 1 \quad \text{if } j = 2, \ldots, n-1,$$

while

$$\lambda(j, \mathscr{D}) = \binom{2n-2}{n-2} \qquad \text{if } j = n, n+1, \ldots, 2n.$$

Thus $\mathscr{D}$ is not a predesign since (ii) of the definition is violated.

Case II. The non-existence of a $(2n, n, \alpha-1, 1)$-predesign when n is not a power of 2.

Suppose a $(2n, n, \alpha-1, 1)$-predesign $\mathscr{D}$ exists. The number of blocks either in $\mathscr{D}$ or complementary to a block in $\mathscr{D}$ is $4\alpha - 4$. The remaining four n-element subsets of V have the form A, A', B, B'. Let $w \in A \setminus B$ and $v \in A \cap B$. The set $\mathscr{P} = \mathscr{D} \cup \{A, B\}$ forms a $(2n, n, \alpha, 1)$-predesign, since $\mathscr{P}$ can obviously be transformed into any given $(2n, n, \alpha, 1)$-design by interchanging some blocks with their complements. Now $\lambda(v, \mathscr{D}) = \lambda(v, \mathscr{P}) - 2$, while $\lambda(w, \mathscr{D}) = \lambda(w, \mathscr{P}) - 1$, so $\mathscr{D}$ can not be a predesign.

Lemma 7. *Suppose the x-element subsets of a k-element set are partitioned into two classes X and Y such that if $S \in X$ and $T \in Y$, then $|S \cap T| \leq x-2$. Then either $X = \emptyset$ or $Y = \emptyset$.*

Proof. If $|A|=x-1$, then for all v, $A\cup\{v\}$ must lie in the same class. We suppose that for all v, $A\cup\{v\}\in X$. We shall show by induction that all x-tuples are in the class X. Let $B\cup\{w\}\in Y$ with $|B\cup A|$ maximum. We know that $|B\cap A|\le x-1$. Let $B\cap A=\{v_1,\dots,v_r\}$, $A=\{v_1,\dots,v_{x-1}\}$, $B=\{v_1,\dots,v_r,w_{r+1},\dots,w_{x-1}\}$ and $B^*=\{v_1,\dots,v_{r+1},w_{r+2},\dots,w_{x-1}\}$. Then $B\cup\{v_{r+1}\}=B^*\cup\{w_{r+1}\}\in X$, contradicting the fact that $B\cup\{w\}\in Y$ for all w.

Lemma 8. *Let $\mathscr{D}$ be a $(2n,n,\frac14\binom{2n}{n},1)$-design. Form a directed graph, $G(\mathscr{D})$, on the vertices V of $\mathscr{D}$ by having an edge (u,v) if there are blocks B_1 and B_2 in $\mathscr{D}$ such that $B_1\cap B_2=\{u\}$ and $B_1\cup B_2=V-\{v\}$. Then $G(\mathscr{D})$ is a graph, that is, $G(\mathscr{D})$ is symmetric and irreflexive.*

Proof. It is a theorem (Theorem 7) of Ehrenfeucht, Faber and Simmons (1979), that every two points u and v occur together in $\frac12\binom{2n-2}{n-1}$ blocks of $\mathscr{D}$ and, consequently, that u occurs in

$$\frac{1}{4}\binom{2n}{n}-\frac{1}{2}\binom{2n-2}{n-2}=\frac{1}{2}\binom{2n-2}{n-1}$$

blocks of $\mathscr{D}$ which omit v. We divide the n-element subsets of $V-v$ into $\frac12\binom{2n-2}{n-1}$ pairs A_1,A_2 such that $A_1\cap A_2=\{u\}$. Since the pair B_1,B_2 are in $\mathscr{D}$, there must be at least one pair A_1,A_2 such that neither A_1 nor A_2 are in $\mathscr{D}$. But then the complements $\bar{A}_1,\bar{A}_2$ must be in $\mathscr{D}$. Since $\bar{A}_1\cap\bar{A}_2=\{v\}$ and $\bar{A}_1\cup\bar{A}_2=V-\{u\}$, (v,u) is an edge in $G(\mathscr{D})$. Since $G(\mathscr{D})$ is obviously irreflexive, this proves the lemma.

Lemma 9. *Suppose $B_1=\{u,x_1,\dots,x_{n-1}\}$ and $B_2=\{u,y_1,\dots,y_{n-1}\}$ are the blocks in $\mathscr{D}$ which join u to v in $G(\mathscr{D})$, that is, $B_1\cup B_2=V-\{v\}$. Then each x_i and y_i is joined either to u or to v in $G(\mathscr{D})$.*

Proof. Consider $(x_1,y_1,\dots,y_{n-1})$. If it is in $\mathscr{D}$, then it and $(u,x_1,\dots,x_{n-1})$ join x_1 to v. Otherwise, $(u,v,x_2,\dots,x_{n-1})\in\mathscr{D}$ and it and $(u,y_1,\dots,y_{n-1})$ join u to x.

Lemma 10. *If $u_1,u_2,\dots,u_r,u_1$ is an r-cycle in $G(\mathscr{D})$, $r=2$ or 3, then $\mathscr{D}$ has a $(2n,n,r,1)$-subdesign.*

Proof. Associated with the cycle $u_1,u_2,\dots,u_r,u_1$ is a collection of pairs of blocks $A_1,B_1,A_2,B_2,\dots,A_r,B_r$ of $\mathscr{D}$ such that $A_i\cap B_i=\{u_i\}$ and $A_i\cup B_i=V-\{u_{i+1}\}$, $i=1,2,\dots,r \pmod r$. Clearly these blocks are all distinct and form a r-tuple cover.

Theorem 11. *Every $(2n,n,\frac14\binom{2n}{n},1)$-design contains a $(2n,n,3,1)$-design.*

Proof. We suppose no triple cover exists in the $(2n,n,\frac14\binom{2n}{n},1)$-design $\mathscr{D}$ and reach a contradiction in a series of steps.

(1) *$G(\mathscr{D})$ is a complete bipartite graph.*

By Lemma 10, $G(\mathscr{D})$ has no triangles. Suppose u is joined to v in $G(\mathscr{D})$. Let $V_1 = \{x \mid (x, v) \in G(\mathscr{D})\}$ and $V_2 = \{y \mid (y, u) \in G(\mathscr{D})\}$. If $x \in V_1$ and $y \in V_2$, by Lemma 9, x is joined to y or to u. If x is joined to u, a triangle is formed by x, u, and v. Thus x is joined to y, so $G(\mathscr{D})$ is complete bipartite.

(2) *There exist V_1 and V_2, subsets of V, with $|V_1| = k$ an odd integer, such that if there exists $B \in \mathscr{D}$ such that $|B \cap V_1| = x$, then no $B' \in \mathscr{D}$ exists such that $|B' \cap V_1| = k - x$.*

Let V_1 and V_2 be the bipartition found in (1). Let $B \cap V_1 = S$, $|S| = x$. Let $X = \{T \subseteq V_2 \mid T \cup S \in \mathscr{D}\}$ and $Y = \{T \subseteq V_2 \mid T \cup S \notin \mathscr{D}\}$. By Lemma 7, if $X, Y \neq \emptyset$, then there exists $T_1 \in X$ and there exists $T_2 \in Y$ such that $|T_1 \cap T_2| > x - 2$. We have $B_1 = T_1 \cup S \in \mathscr{D}$ and $B_2 = T_2 \cup S \notin \mathscr{D}$. Thus $(V_2 - T_2) \cup (V_1 - S) = B_2' \in \mathscr{D}$. We find that $B_1 \cap B_2' = T_1 \setminus T_2$ and $V \setminus (B_1 \cup B_2') = T_2 \setminus T_1$. These sets can't be empty for if $T_1 = T_2$, then $B_1 = B_2$ is both in $\mathscr{D}$ and not in $\mathscr{D}$. Thus $|T_1 - T_2| = |T_2 - T_1| = 1$, which contradicts the fact that no point of V_2 is joined to any other. Thus for each $S \subseteq V_1$ with $|S| = x$ either

(i) $S \cup T \in \mathscr{D}$ for all $T \subseteq V_2$ and $|T| = n - x$, or

(ii) $S \cup T \notin \mathscr{D}$ for all $T \subseteq V_2$ and $|T| = n - x$.

The symmetrical argument with V_1 and $n - x$ shows either

(i) $S \cup T \in \mathscr{D}$ for all $S \subseteq V_1$ and $|S| = x$, or

(ii) $S \cup T \notin \mathscr{D}$ for all $S \subseteq V_1$ and $|S| = x$.

Thus for each x, $0 \leq x \leq n$, either

(i) $S \cup T \in \mathscr{D}$ for all $S \subseteq V_1$, $T \subseteq V_2$, $|S| = x$, $|T| = n - x$, or

(ii) $S \cup T \notin \mathscr{D}$ for all $S \subseteq V_1$, $T \subseteq V_2$, $|S| = x$, $|T| = n - x$.

To show that $|V_1| = k$ is odd, suppose $k = 2r$. Then with $x = r$ (working with the complementary design if necessary), we have that there exists $S \subseteq V_1$ with $|S| = r$ such that $S \cup T \in \mathscr{D}$ with $|T| = n - r$. Also since $|V_1 - S| = r$ and $|V_2 - T| = n - r$, we have $V - (S \cup T) = (V_1 - S) \cup (V_2 - T) \in \mathscr{D}$, contradicting the fact that every two blocks of $\mathscr{D}$ meet.

(3) *Let $k = 2r + 1 \leq n$ and let*

$$S = S(n, r) = \sum_{i=0}^{2r} (-1)^i \binom{2r}{i} \binom{2n - 2r - 1}{n - i - 1}.$$

We must have $S = 0$.

Note that by (2), if there exists $B \in \mathscr{D}$ such that $|B \cap V_1| = x$, every x-tuple in V_1 occurs as $V_1 \cap A$ for some block $A \in \mathscr{D}$. It is not possible that there exists a block B for each $1 \leq t \leq r + 1$ such that $|V_1 \cap B| = r + t$, for then an element $v \in V_1$ appears in

$$\sum_{t=1}^{r+1} \binom{2r}{r + t - 1} \binom{2n - 2r - 1}{n - r - t} = \sum_{s=r}^{2r} \binom{2r}{s} \binom{2n - 1 - 2r}{n - 1 - s}$$
$$> \frac{1}{2} \sum_{s=0}^{2r} \binom{2r}{s} \binom{2n - 1 - 2r}{n - 1 - s} = \frac{1}{4} \binom{2n}{n}$$

blocks, an impossibility. We shall say that the number x is *used* by the design $\mathscr{D}$ if $|V_1 \cap B| = x$ for some $B \in \mathscr{D}$. We know that one and only one of each pair x or $2r + 1 - x$ is used by $\mathscr{D}$ for $1 \leq x \leq r$, and not all $x \geq r + 1$ can be used. By considering the complementary design, if necessary, we may suppose $r + 1$ is used by $\mathscr{D}$. If $r + i - 2$ and $r - i + 1$, for some $i \geq 3$, are both used by $\mathscr{D}$, we can make a triple cover as follows.

We construct 6 subsets of V_1, called $S_1, S_2, \ldots, S_6$ with $|S_1|=|S_2|=|S_3|=|S_4|=r+1$, $|S_5|=r+i-2$ and $|S_6|=r-i+1$ by placing each element of V_1 in 3 of the sets until the sets have the proper cardinality (which elements go where is of no consequence). Since $4(r+1)+(r-i+1)+(r+i-2)=3(2r+1)$, each element of V_1 is covered 3 times. Similarly, subsets $T_1, T_2, \ldots, T_6$ of V_2 are formed so that each element of V_2 is covered thrice and $|T_1|=|T_2|=|T_3|=|T_4|=n-r-1$, $|T_5|=n-(r+i-2)$, $|T_6|=n-(r-i+1)$. The blocks $S_i \cup T_i$, $i=1,2,\ldots,6$ form a triple cover. This shows that if $r+i-2$ is used, then $2r+1-(r-i+1)=r+i$ must be used. Since $r+1$ is used, $r+3, r+5, \ldots$ must be used. Let $j \geq 1$ be the smallest such that $r+2j$ is used. As before, $r+2j+2$, $r+2j+4, \ldots, 2r$ must be used. Thus $j \neq 1$. This means that $r-1$ is used. Now consider the greatest of the series $r-3, r-5, \ldots$ which is not used. Call this $r-i+1$. Then $r+i$ and $r-i+3$ must be used. Since $2(r+1)+2(r-1)+(r+i)+(r-i+3)=3(2r+1)$, a triple cover can be constructed as before. The only possibility is that $r+1, r+3, \ldots, r-1, r-3, \ldots$ are the integers which are used. The number of times $v \in V_1$ is covered in $\mathscr{D}$ is

$$\cdots+\binom{k-1}{r-2}\binom{2n-k}{n-r-1}+\binom{k-1}{r}\binom{2n-k}{n-r-1}+\binom{k-1}{r+2}\binom{2n-k}{n-r-3}+\cdots.$$

The number of times v is *not* covered is

$$\cdots+\binom{k-1}{r-3}\binom{2n-k}{n-r+2}+\binom{k-1}{r-1}\binom{2n-k}{n-r}+\binom{k-1}{r+1}\binom{2n-k}{n-r-2}+\cdots.$$

Since these two values must be equal, we must have

$$S=\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}\binom{2n-k}{n-i-1}=0.$$

(4) *Let S be as in* (3). *Then $S \neq 0$.*
Consider

$$\begin{aligned}&\binom{2r}{i}\binom{2n-2r-1}{n-i-1}+(-1)^{2r-(i+1)}\binom{2r}{2r-i-1}\binom{2n-2r-1}{n-1-(2r-i-1)}\\&=\binom{2r}{i}\binom{2n-2r-1}{n-i-1}-\binom{2r}{i+1}\binom{2n-2r-1}{n-i-1}\\&=\left[\binom{2r}{i}-\binom{2r}{i+1}\right]\binom{2n-2r-1}{n-i-1}.\end{aligned}$$

We write

$$\begin{aligned}S&=\sum_{i=0}^{r-1}(-1)^i\left[\binom{2r}{i}-\binom{2r}{i+1}\right]\binom{2n-2r-1}{n-i-1}+(-1)^{2r}\binom{2n-2r-1}{n-2r-1}\\&=\sum_{i=0}^{r-1}(-1)^{i+1}\binom{2r+1}{i+1}\binom{2n-2r-1}{n-i-1}+\binom{2n-2r-1}{n-2r-1}.\end{aligned}$$

Let

$$S_i = \binom{2r+1}{i+1}\binom{2n-2r-1}{n-i-1} - \binom{2r+1}{i}\binom{2n-2r-1}{n-i}.$$

It is easily seen that $S_i > 0$ if $i \le r-1$. Then

$$S = S_1 + S_3 + \cdots + S_{r-1} + \binom{2n-2r-1}{n-2r-1} > 0$$

when r is even, and

$$S = -S_0 - S_2 - \cdots - S_{r+1} < 0$$

when r is odd.

This contradiction proves the theorem.

Theorem 12. *Every* $(2n, n, \alpha, 1)$*-design, where* $\alpha = \frac{1}{4}\binom{2n}{n}$*, contains at least*

$$k = \left[\frac{1}{2(2n-1)}\alpha\right]$$

disjoint double covers, and hence contains $(2n, n, 2\lambda, 1)$*-designs for all* $\lambda \le k$*.*

Proof. Let $\mathscr{D}$ be a $(2n, n, \alpha, 1)$-design on V and let $v \in V$. Consider the α subsets $S \subseteq V$ of size $n-1$ such that $S \cup \{v\}$ is a block in $\mathscr{D}$. By the Erdös-Ko-Rado Theorem (1961), there is a pair S_1, T_1 in this collection such that $S_1 \cap T_1 = \emptyset$ since $\alpha > \binom{2n-2}{n-2}$. In fact, if the pairs S_i, T_i, $i = 1, 2, \dots, k-1$, are found with the property $S_i \cap T_i = \emptyset$ and removed from the collection, there will still remain a pair S_k, T_k with $S_k \cap T_k = \emptyset$, as long as

$$\alpha - 2(k-1) > \binom{2n-2}{n-2}.$$

This relation can be easily seen to be satisfied if $k \le \alpha/2(2n-1)$. Let $A_i = \{v\} \cup S_i$ and $B_i = \{v\} \cup T_i$, $i = 1, 2, \dots, k$, where $k = [\alpha/2(2n-1)]$. Now suppose $u \in V \setminus \{v\}$ and S_i, T_i for $1 \le i \le r$ have the property that they are the only pairs for which $u \notin S_i \cup T_i$. By the argument used in the proof of Lemma 8, there must be at least r pairs Q_i, R_i of $n-1$ element subsets of V such that $Q_i \cup R_i = V - \{u, v\}$ and neither $\{v\} \cup Q_i$ nor $\{v\} \cup R_i$ are blocks of $\mathscr{D}$, $i = 1, 2, \dots, r$. But then $C_i = \{u\} \cup Q_i$ and $D_i = \{u\} \cup R_i$ are blocks in $\mathscr{D}$. In this way, the blocks A_i, B_i, C_i, D_i form a double cover for each $i = 1, 2, \dots, k$. This proves the theorem.

Theorem 13. *Every* $(2n, n, \lambda, 1)$*-design* $\mathscr{D}$ *with*

$$\lambda > \left(1 - \frac{1}{2n-1}\right)\frac{1}{4}\binom{2n}{n} + 2(n-1)$$

contains a $(2n, n, \gamma, 1)$*-design for some* γ *with* $2 \le \gamma \le 2n$*, and hence is not primitive.*

Proof. Suppose a sequence of distinct points v_i, $i=1,2,\ldots,k+1$, and a sequence of distinct blocks A_i, B_i, $i=1,2,\ldots,k$, of the design $\mathscr{D}$ have been found with the properties $A_i \cap B_i = \{v_i\}$ and $V-(A_i \cup B_i)=\{v_{i+1}\}$. The point v_{k+1} is a member of at least $\lambda-k+1$ blocks of $\mathscr{D}$ distinct from $\{A_i, B_i\}$, since v_{k+1} is a member of at most one of A_i and B_i for $i=1,2,\ldots,k-1$ and neither A_k nor B_k contains v_{k+1}. If $\lambda-k+1>\binom{2n-2}{n-2}$, The Erdös–Ko–Rado Theorem yields blocks A_{k+1}, B_{k+1} in $\mathscr{D}$, distinct from $\{A_i, B_i \mid i\le k\}$, such that $A_{k+1}\cap B_{k+1}=\{v_{k+1}\}$ (see the proof of Theorem 12). This inequality can be written equivalently as

$$\lambda > \left(1-\frac{1}{2n-1}\right)\frac{1}{4}\binom{2n}{n}+k-1,$$

so by assumption, it holds for all $k\le 2n-1$. Since there are only $2n$ points in V, we can find a sequence of distinct points v_i, $i=1,2,\ldots,\gamma$, and a sequence of distinct blocks A_i, B_i, $i=1,2,\ldots,k$, such that $A_i\cap B_i=\{v_i\}$ and $V-(A_i\cup B_i)=\{v_{i+1}\}$ for all i, where $v_{\gamma+1}=v_1$. This collection of blocks forms a $(2n,n,\gamma,1)$-design.

Theorem 14. *Every $(2n,n,\lambda,1)$-design with $\lambda\le\frac{1}{2}\frac{1}{4}\binom{2n}{n}-2$ can be extended to a $(2n,n,\lambda+2,1)$-design.*

Proof. Let $\alpha=\frac{1}{4}\binom{2n}{n}$. Let $\mathscr{D}$ be a $(2n,n,\lambda,1)$-design on V. Let $\mathscr{S}=\{A_i, A_i' \mid i\le 2\alpha-2\lambda\}$ be the n-element subsets of V which are neither blocks in $\mathscr{D}$ nor complements of blocks in $\mathscr{D}$. Each element of V is covered exactly $2\alpha-2\lambda$ times by the blocks in $\mathscr{S}$. The average over all pairs $(v.w)$, $v\ne w$, of the number of blocks in $\mathscr{S}$ which contain v and miss w is

$$\frac{n}{2n-1}(2\alpha-2\lambda).$$

There must be one pair (v,w) for which this number is at least as great as the average. We wish to find two pairs of blocks A_1, B_1 and A_2, B_2 such that

$$A_1\cap B_1=\{v\}=A_2\cap B_2 \quad\text{and}\quad V-(A_1\cup B_1)=\{w\}=V-(A_2\cup B_2).$$

This can be accomplished by the Erdös–Ko–Rado Theorem (see the proof of Theorem 12) since $\lambda\le\frac{1}{2}\alpha-2$ implies that $\lambda<\frac{1}{2}\alpha-2+1/n$ which can be rearranged to yield

$$\frac{n}{2n-1}(2\alpha-2\lambda)>\binom{2n-3}{n-2}+2.$$

We extend $\mathscr{D}$ by adding the double cover $\{A_1, B_1, A_2', B_2'\}$ to it.

Problem 1. What are necessary and sufficient conditions such that a $(2n,n,\lambda,1)$-predesign can be transformed into a $(2n,n,\lambda,1)$-design by repetition of the operation of interchanging some block in the design with its complement?

Remark 2. This is not always possible (Simmons, 1981).

Problem 2. How large can a primitive $(2n, n, \lambda, 1)$-design be?

Problem 3. Which $(2n, n, \lambda, 1)$-design can be extended?

Problem 4. We know that if λ is large enough, a $(2n, n, \lambda, 1)$-design is not primitive, but does such a design always contain a double cover?

Problem 5. What happens if we insist that every two blocks meet in at least 2 points?

Problem 6. What can be said about general projective $(n, k, \lambda, 1)$-designs?

Remark 3. Bollobás and Erdös conjectured and Lovász (1978) proved that projective $(n, k, \lambda, 1)$-designs exist only if $k^2 - k + 1 \geq n$.

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