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RAMSEY-MINIMAL GRAPHS FOR FORESTS

Stefan A. BURR

City College, CUNY, New York, NY 10036, USA

P. ERDÖS

Hungarian Academy of Sciences, Budapest, Hungary

R.J. FAUDREE, C.C. ROUSSEAU, R.H. SCHELP

Memphis State University, Memphis, TN 38152, USA

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It is shown in this paper that the pair (G, H) is Ramsey infinite when both G and H are forests, with at least one of G or H having a non-star component. In addition, an infinite subfamily of $\mathcal{R}(P_n, P_n)$ is constructed.

1. Introduction

Let F, G, and H be graphs (no loops or multiple edges). We write $F \to (G, H)$ if whenever each edge of F is colored red or blue, then either the red subgraph of F, denoted $(F)_R$, contains a copy of G or the blue subgraph of F, denoted $(F)_B$, contains a copy of H. The graph F is called *(G, H)-minimal* if $F \to (G, H)$ and $F' \not\to (G, H)$ for each proper subgraph F' of F. In particular, if G, H, and F have no isolated vertices, F' can be replaced by $F - e$, where e is any edge of F. The class of all (G, H)-minimal graphs will be denoted by $\mathcal{R}(R, H)$. The pair (G, H) will be called *Ramsey-finite* or *Ramsey-infinite* depending upon whether $\mathcal{R}(G, H)$ is finite or infinite.

This paper is essentially a continuation of [3], where $\mathcal{R}(G, H)$ is considered for H and G both star-forests, i.e., forests of stars. There it is shown that if G and H are star-forests with no single-edge stars, then (G, H) is Ramsey-finite if and only if both G and H are single stars with an odd number of edges. The case when G or H have some single-edge stars is not completely answered. Some particular cases are considered with the general question of the finiteness of $\mathcal{R}(G, H)$, when either G or H contain single-edge stars, left open.

There are other papers which discuss similar problems. In particular see [2, 4, 5, 6, 9]. Nešetřil and Rödl proved that (G, H) is Ramsey-infinite if both G and H are 3-connected [10], or if both G and H are at least 3-chromatic [10], or if G and H are forests, neither of which is a star-forest [9]. In light of [9] and [3] this leaves the case when one of G or H is a star-forest and the other is a forest with at least one non-star component. The central result (Theorem 7) of the next section settles this case. It is shown in [4] that (G, H) is Ramsey-finite when G is a matching and H is any graph. In addition if (G, H) is Ramsey-finite for each graph H, then the results of [5] indicate G must be matching.

We need to introduce some further notation and terminology. The word "coloring" will always refer to coloring each edge of some graph red or blue. A coloring of F with neither a red G or a blue H will be called *(G, H)-good* or, if the meaning is clear, simply a *good coloring*. If G is a subgraph of $(F)_R$ this may be denoted by $G \leq (F)_R$. For the graph G, $V(G)$ and $E(G)$ will denote its vertex and edge sets respectively. For typographical reasons the star $K_{1,n}$ will be symbolized by S_n. As in [6], a *(G, H, γ)-determiner* will be a graph which has (G, H)-good colorings, but only ones in which the edge γ is red. One could call such a graph a "red" determiner, but for compactness of notation we will not do so, since a (H, G, γ)-determiner is the same as a "blue" determiner. Naturally, the reader must be careful to observe the distinction. Also for compactness, we will drop the γ or even the G and H when the meaning is clear. In a (G, H, γ)-determiner the edge γ will be called the *determined edge*.

We will sometimes need stronger types of determiners. A well-behaved determiner is one which has good colorings in which the determined edge is red, but all adjacent edges are blue. A (G, H, γ)-determiner is *minimal* if no proper subgraph of it is a (G, H, γ)-determiner. Observe that a minimal determiner contained in a well-behaved one is also well-behaved.

In what follows, we will frequently construct graphs by identifying vertices or edges of other graphs. The reader is to understand that in such cases, all vertices and edges remain distinct, except for those explicitly made the same by the identifications specified.

Further notation and terminology will follow that of [1] and [8].

2. Main results

The reader should observe that in several of the theorems to follow, we use the following simple condition which is equivalent to that of the pair (G, H) being Ramsey-infinite: for each positive integer n_0 there exists a graph in $\mathcal{R}(G, H)$ with at least n_0 vertices. The following lemma is an example of the methods used to establish such a condition.

Lemma 1. *Let T and U be trees with at least three vertices. If there exists a well-behaved minimal (T, U, α)-determiner where α is a free edge, then there exists a well-behaved minimal (U, T, β)-determiner where β is a free edge, which has more vertices.*

Proof. Let H be a well-behaved minimal (T, U, α)-determiner with α a free edge. (Recall that a free edge is one with a vertex of degree one.) Let H' be formed from H by removing the vertex of degree 1 from α, and denote the other vertex of α by v. Also, let x and y denote two distinct end vertices of T. Let x' and y' be the neighbors of x and y respectively; x' and y' need not be distinct. Now take a copy T^* of T; for each vertex of T^*, except x, x', and y', take a copy of H' and identify its v with that vertex of T^*. Call the resulting graph L and let β be the edge xx'. We will show that this graph is a well-behaved (U, T, β)-determiner, and moreover that any (U, T, β)-determiner contained in L has more points than H.

To see this, observe that in any (U, T)-good coloring of L, all edges of T^* which are incident to points corresponding to v in a copy of H' must be blue, since each H' must be (U, T)-good colored. That is, all edges of T^*, other than β, must be blue; hence β must be red. Furthermore, it is clear that good colorings of this type exist, so that L is a determiner. In addition, we see from these colorings that L is well-behaved. Now delete edges from L until it becomes a minimal determiner, and consider the edges of T^*; none of these could have been removed, for then β could be colored blue in some (U, T)-good coloring. Similarly, no edge of the copy of H' attached at y could have been removed, for then the edge yy' could be colored red and β colored blue in some good coloring. Since any subgraph of L containing all of this copy of H' and all of T^* has more vertices than H, the proof is complete.

Theorem 2. *Let T_n be a tree on n vertices which is not a star. Then (S_k, T_n) is Ramsey-infinite if and only if $k \geq 2$.*

Proof. Since $\mathcal{R}(S_1, G) = \{G\}$ for any graph G, we need only to show that (S_k, T_n) is Ramsey-infinite when $k \geq 2$. Our first step (the biggest one) is to show the existence of a (S_k, T_n, α)-determiner when $k \geq 2$. This determiner will not in general be well-behaved.

Consider a K_{n-1} and label a fixed vertex v. If $k \geq 2$, then attach to this K_{n-1} at v a S_{k-2}, by identifying the central vertex of the star with the vertex v, so that the star is otherwise disjoint from K_{n-1}. At each of the remaining $n-2$ vertices of K_{n-1} attach a S_{k-1} by its central vertex. This constructed graph, which we call $J(v)$, has $(n-2)k + k - 1$ vertices, $n-2$ of them of degree $n+k-3$, one of them (namely v) of degree $n+k-4$, and the remaining $(n-2)(k-1)+(k-2)$ of degree 1.

If L is a graph with some vertices of degree 1, define $O(L)$ to be that graph obtained from L by attaching to each of its vertices of degree 1 a different copy of $J(v)$, identifying v with the vertex of degree 1. Note that if L has t vertices of degree 1, then $|V(O(L))| = |V(L)| + t(|V(J(v))| - 1)$.

Define $H_1 = J(v)$ and let $H_{i+1} = O(H_i)$ for $i = 1, 2, \ldots, n-1$. Set $H' = H_n$ and let v_0 denote the vertex v of H' which is also in H_1. Take an edge $\alpha = \{x, y\}$ which is vertex-disjoint from H' and form the graph H^* by attaching α to H' at v_0, identifying y with v_0.

We show H^* is a (S_k, T_n, α)-determiner. To see this we first give a good coloring to H^* with α colored red. For such a coloring, color each copy of $J(v)$ contained in H^* as follows: color all the edges of the K_{n-1} in $J(v)$ blue and all edges of all stars attached to the K_{n-1} in $J(v)$ red. This colors all edges of H^* but α, which we also color red. Clearly this gives a good coloring of H^*. To see that under all (S_k, T_n)-good colorings of H^* edge α must be red, suppose the contrary, giving H^* a good coloring with α blue. Observe that all vertices of H^* (except for x) that are at distance $n-1$ or less from v_0 are of degree $n+k-3$. Hence these vertices are of degree $n-2$ or more in $(H^*)_B$. Delete an end-vertex u from T_n with $\{w, u\}$ an edge of T_n and let T' represent this tree on $n-1$ vertices. Clearly since all vertices of $(H^*)_B$, different from x, within a distance $n-1$ or less from v_0 are at least of degree $n-2$, $T' \leq (H^*)_B - \alpha$ with T' rooted with root w at v_0. But edge α is blue, giving $T_n \leq (H^*)_B$, a contradiction. Hence H^* is a (S_k, T_n, α)-determiner, which we denote as $H^*(\alpha)$.

We now use this (S_k, T_n, α)-determiner to construct a well-behaved minimal (T_n, S_k, β)-determiner. Take $k-1$ copies of $H^*(\alpha)$, and identify the end vertices of the $k-1$ edges corresponding to α; designate this vertex by z. Now attach a free edge β at z. Clearly, β is red in any (T_n, S_k)-good coloring of this graph. In addition, it has a (T_n, S_k)-good coloring in which β is red and all edges adjacent to it (the copies of α) are blue. Hence it is a well-behaved (T_n, S_k, β)-determiner. Remove edges to form a minimal one, which is clearly also well-behaved.

Now we invoke Lemma 1 enough times to form an arbitrarily large well-behaved minimal (S_k, T_n, γ)-determiner. Take our well-behaved minimal (T_n, S_k, β)-determiner (or any other such), and identify β with γ, keeping the original end vertices of β and γ distinct; call the resulting graph F. This is our desired large (S_k, T_n)-minimal graph, since clearly $F \to (S_k, T_n)$, and if an edge of the (S_k, T_n, γ)-determiner is removed, there will be a good coloring in which $\gamma (=\beta)$ is blue. This completes the proof.

Notice that each edge of F is part of a T_n; otherwise color this edge (call it δ) blue and give $F-\delta$ an (S_k, T_n)-good coloring, which results in a (S_k, T_n)-good coloring of F. Hence since F has large diameter it has "many" disjoint copies of T_n. This gives the following corollary to the theorem.

Corollary 3. *Let T_n be a tree on n vertices which is not a star. Then for each fixed l and for $k \geq 2$ the pair $(S_k \cup lS_1, T_n)$ is Ramsey-infinite.*

We will prove two useful general theorems whose proofs are similar. One of these results is used in the main theorem (Theorem 7) of this section. First, however, we prove a lemma.

Lemma 4. *Let $\{G_i\}_{1\le i\le m}$ and $\{H_j\}_{1\le j\le n}$ be families of connected graphs. Let $F_{ij}\in\mathcal{R}(G_i,H_j)$ for each i and j. Then there exists a subcollection $\mathcal{G}$ of the family $\{F_{ij}\}_{1<i<m,\,1\le j\le n}$ such that*

(1) *for each i and j there exists an $L\in\mathcal{G}$ such that $L\to(G_i,H_j)$, and*

(2) *there exists a fixed s and t such that $F_{st}\in\mathcal{G}$ and $L\not\to(G_s,H_t)$ when $L\in\mathcal{G}-\{F_{st}\}$.*

Proof. We will actually prove this result for any subset of the set of pairs $\{(G_i,H_j)\}$. The proof will be by induction on the number of such pairs in the subset.

Clearly the result holds when there is one ordered pair. Thus assume there are $k+1$ ordered pairs of graphs (G_i,H_j) and that the result holds when there are k. Take any k of these $k+1$ pairs. By assumption there exists a subcollection $\mathcal{G}'$ of these k ordered pairs such that both (1) and (2) hold. Consider the remaining pair $(G_{i'},H_{j'})$ and its arrowing graph $F_{i'j'}$. If there exists an $L\in\mathcal{G}'$ such that $L\to(G_{i'},H_{j'})$, then set $\mathcal{G}=\mathcal{G}'$. Otherwise take $\mathcal{G}=\mathcal{G}'\cup\{F_{i'j'}\}$. Clearly $\mathcal{G}$ as defined satisfies the conditions of the lemma and the proof is complete.

Before stating the next theorem we introduce some additional terminology. Let $\{G_i\}_{1\le i\le m}$ and $\{H_j\}_{1\le j\le n}$ be families of connected graphs. Let $\mathcal{R}(\bigwedge_{i=1}^m G_i,\bigwedge_{j=1}^n H_j)$ denote those graphs which when colored either contain red copies of G_i for each i, $1\le i\le n$, or blue copies of H_j for each j, $1\le j\le n$, but each proper subgraph can be given a $(\bigwedge_{i=1}^m G_i,\bigwedge_{j=1}^n H_j)$-good coloring. Here a $(\bigwedge_{i=1}^m G_i,\bigwedge_{j=1}^n H_j)$-good coloring of a graph means that the graph can be colored so there exist a fixed i and j such that the graph contains no red G_i and no blue H_j. Also, we give the term "Ramsey-infinite" the obvious meaning in this case.

Theorem 5. *Let $\{G_i\}_{1\le i\le m}$ and $\{H_j\}_{1\le j\le n}$ be families of connected graphs. If (G_i,H_j) is Ramsey-infinite for each i and j, then $(\bigwedge_{i=1}^m G_i,\bigwedge_{j=1}^n H_j)$ is Ramsey-infinite.*

Proof. Let n_0 be a fixed positive integer. Since $\mathcal{R}(G_i,H_j)$ is infinite for all i and j, pick $F_{ij}\in\mathcal{R}(G_i,H_j)$ such that $|V(F_{ij})|>n_0$ for all i and j. By Lemma 4 there exists a subcollection $\mathcal{G}$ of $\{F_{ij}\}_{1<i<m,\,1\le j\le n}$ such that (1) for each i and j there exists an $L\in\mathcal{G}$ such that $L\to(G_i,H_j)$ and (2) there exists a fixed s and t such that $F_{st}\in\mathcal{G}$ and $L\not\to(G_s,H_t)$ when $L\in\mathcal{G}-\{F_{st}\}$. Thus we have that $\bigcup\mathcal{G}\to(\bigwedge_{i=1}^m G_i,\bigwedge_{j=1}^n H_j)$ and if M is a subgraph of $\bigcup\mathcal{G}$ such that $M\in\mathcal{R}(\bigwedge_{i=1}^m G_i,\bigwedge_{j=1}^n H_j)$, then M contains F_{st} as a subgraph. But $|V(F_{st})|>n_0$ so that $|V(M)|>n_0$. Hence $(\bigwedge_{i=1}^m G_i,\bigwedge_{j=1}^n H_j)$ is Ramsey-infinite.

Theorem 6. *Let $\{G_i\}_{1\le i\le m}$ and $\{H_j\}_{1\le j\le n}$ be families of connected graphs. If (G_i,H_j) is Ramsey-infinite for each i and j, then $(\bigcup_{i=1}^m G_i,\bigcup_{j=1}^n H_j)$ is Ramsey-infinite.*

Proof. As in the previous proof, let n_0 be a fixed positive integer. Pick $F_{ij} \in \mathcal{R}(G_i, H_j)$ such that $|V(F_{ij})| > n_0$ for all i and j. Choose $\mathcal{G}$ as in Lemma 4 and let $F = \bigcup \mathcal{G}$. Set $v = m + n - 1$; clearly $vF \to (\bigcup_{i=1}^m G_i, \bigcup_{j=1}^n H_j)$. Also since $F - \{F_{st}\} \not\to (G_s, H_t)$, it follows that a subgraph M of vF such that $M \in \mathcal{R}(\bigcup_{i=1}^m G_i, \bigcup_{j=1}^n H_j)$ must contain F_{st} as a subgraph. Hence $|V(M)| > n_0$ and the result follows.

We now prove our main result.

Theorem 7. *Let G and H be forests such that neither forest is a matching and at least one of the forests has a component which is not a star. Then the pair (G, H) is Ramsey-infinite.*

Proof. The case when both G and H have components which are not stars has been proved by Nešetřil and Rödl in [9]. Thus we may assume that

$$G = S_{m_1} \cup S_{m_2} \cup \cdots \cup S_{m_w} \cup qS_1$$

and

$$H = S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_s} \cup T_1 \cup T_2 \cup \cdots \cup T_z,$$

where $m_1 \geq m_2 \geq \cdots \geq m_w \geq 2$, $n_1 \geq n_2 \geq \cdots \geq n_s \geq 1$, $z \geq 1$, and each T_i is a non-star tree.

Let n_0 be a fixed positive integer. Choose $M \in \mathcal{R}(\bigcup_{i=1}^w S_{m_i}, \bigcup_{j=1}^z T_j)$ such that (setting $G_i = S_{m_i}$ and $H_j = T_j$ for all i and j) M is one of the graphs constructed in the proof of Theorem 6. Recall that M is a subgraph of vF, where $v = z + w - 1$ and $F = \bigcup \mathcal{G}$. But $\mathcal{G}$ is a subcollection of $\{F_{ij}\}_{1 \leq i \leq w,\, 1 \leq j \leq z}$ where $F_{ij} \in \mathcal{R}(S_{m_i}, T_j)$. Further it can be assumed that each F_{ij} is one of the graphs constructed in Theorem 2 and that $|V(F_{ij})| > n_0$ with $F_{ij} \to (S_{m_i} \cup qS_1, T_j)$ for all i and j. The assumption that $F_{ij} \to (S_{m_i} \cup qS_1, T_j)$ follows as in Corollary 3. Thus, since $F_{st} \leq M$ for some s and t, not only is $M \in \mathcal{R}(\bigcup_{i=1}^w S_{m_i}, \bigcup_{j=1}^z T_j)$ but also $M \in \mathcal{R}(G, \bigcup_{j=1}^z T_j)$.

It is clear that $(s + w + q - 1)S_{m_1+n_1-1} \cup M \to (G, H)$. Furthermore, for each proper subgraph M' of M, the graph $(s + w + q - 1)S_{m_1+n_1-1} \cup M'$ can be (G, H)-good colored. Just color all edges of $(s + w + q - 1)S_{m_1+n_1-1}$ blue and give M' a $(G, \bigcup_{j=1}^z T_j)$-good coloring. Hence, since $|V(M)| > n_0$, if M^* is a subgraph of $(s + w + q - 1)S_{n_1+m_1-1} \cup M$ such that $M^* \in \mathcal{R}(G, H)$ we have that $|V(M^*)| > n_0$. This establishes the desired result.

We next prove a general result which will be useful elsewhere in showing that (G, H) is Ramsey-infinite for certain graphs G and H. The proof of this result is similar to those of Lemma 1 and Theorem 2, so we will be somewhat brief.

Theorem 8. *Let G be a graph with connectivity at least two and T a tree with at* *least three vertices. If there exists a (G, T, α)-determiner, where α is a free edge, then (G, T) is Ramsey-infinite.*

Proof. We begin by constructing a (T, G, β)-determiner. Let H be a minimal (G, T, α)-determiner with α a free edge. Let H' be formed from H by removing the end vertex of α, denoting the other vertex of α by v. Take a copy of G, calling one of its edges β. For every vertex of G not on β, take a copy of H' and identify its vertex corresponding to v with that vertex of G. Call the resulting graph J. It is easy to see that this is a well-behaved (T, G, β)-determiner. It turns out that any minimal (T, G, β)-determiner contained in J is larger than H, but we do not use this fact. Instead, we must show the existence of a (G, T, γ)-determiner that is larger than H.

Now take a copy of T with distinct free edges γ and δ; let x be the end vertex on δ. We will use this T as the basis for a (G, T, γ)-determiner. For every edge of this T, other than γ, take a copy of J and identify its determined edge with that edge of T. Call the resulting graph F. It is easily seen that this is a (G, T, γ)-determiner. Consider now the vertex x and let xy be an edge other than δ. This edge is an edge of a copy of G on which a J was based; therefore, in this J, y has an H' rooted at it.

Consider the effect of removing any edge of this H'. It would then be possible to give the copy of J in question a good coloring in which xy is blue, but all other edges of the G it is based on (including δ) are red. One could then give all the other J their usual good colorings, and could then color γ blue, since δ was red. We conclude that any minimal (G, T, γ)-determiner F_1 contained in F must leave a copy of H', and of course the T on which it is based, intact. Hence we have that F_1 has more vertices than H.

We now iterate this process enough times to form an arbitrarily large minimal (G, T, γ)-determiner F'. Also take any well-behaved (T, G, β)-determiner F'' and identify β and γ; call the resulting graph F^*. Clearly $F^* \to (G, T)$. Furthermore, it is easy to see that if e is any edge in F', $F^* - e \not\to (G, T)$. This yields the theorem immediately.

Lemma 9. *Let T_n be a tree on n vertices, $n \geq 2$, and let m be a positive integer, $m \geq 2$. Then the only (K_m, T_n)-good coloring of $K_{(m-1)(n-1)}$ has $(K_{(m-1)(n-1)})_B = (m-1)K_{n-1}$.*

Proof. The proof is by induction on m; the result is clear for $m = 2$ and each fixed n. Thus assume the result holds for all positive integer values less than a fixed m and for all values of n.

Give $L = K_{(m-1)(n-1)}$ a (K_m, T_n)-good coloring. Let T' be a vertex-maximal subtree of T_n such that $T' \leq (L)_B$. Since $K_{(m-1)(n-1)}$ is good-colored, $|V(T')| \leq n-1$. Select a vertex v of T' such that if a free edge $e = \{x, y\}$ is attached to T' at v, the resulting tree is still a subtree of T_n. Thus each edge of L incident to v but not in T' is red. This means, since L has been good-colored, that this coloring includes a (K_{m-1}, T_n)-good coloring on each $(m-2)(n-1)$-element subgraph of $\langle V(L)-V(T')\rangle$. Let A be such a subgraph. By the induction assumption $(A)_B = (m-2)K_{n-1}$. Thus each edge of L, not in $A \cup T'$ but incident to a vertex of A, must be red. But then $|T'| = n-1$ and $\langle V(L)-V(A)\rangle = K_{n-1} \leq (L)_B$, so that the result follows.

Lemma 10. *Let T_n be a tree on n vertices $n \geq 2$ and let m be a positive integer, $m \geq 3$. Then there exists a (K_m, T_n, γ_1)-determiner with determined edge γ_1 a free edge.*

Proof. Take a $K_{(m-1)(n-1)}$ and attach a free edge γ_1 to it. By Lemma 9 the resulting graph is clearly a (K_m, T_n, γ_1)-determiner.

Corollary 11. *Let T_n be a tree on n vertices, $n \geq 3$, and let m be a positive integer, $m \geq 3$. Then (K_m, T_n) is Ramsey-infinite.*

Proof. By Lemma 10 there exists a (K_m, T_n, γ_1)-determiner with determined edge γ_1 a free edge. Thus the result follows from Theorem 8.

3. An infinite subcollection of $\mathcal{R}(P_n, P_n)$

As was pointed out in the proof of Theorem 6, Nešetřil and Rödl [9] proved that (F_1, F_2) is Ramsey-infinite when each F_i is a forest containing a non-star component. Their method of proof, although straightforward and elegant, suffers from being rather nonconstructive in nature. Although their method permits in principle one to find arbitrarily many members of $\mathcal{R}(F_1, F_2)$ by an exhaustive search, the amount of work grows without limit. In particular, no infinite class can be actually exhibited by their method. For this reason, we will give a method for exhibiting infinitely many members of $\mathcal{R}(P_n, P_n)$, where P_n is a path on n vertices. For any fixed n, only a finite amount of work is needed to construct an entire infinite subset of $\mathcal{R}(P_n, P_n)$. For small n, this could certainly be done explicitly, but we will not do so. It would be desirable, of course, to carry this one step further and directly construct such subsets for all n simultaneously.

To shorten the presentation two of the theorems will not be proved, although enough information will be given that the interested reader will be able to supply the proofs. It should be noted that the constructions given will not work for $\mathcal{R}(P_m, P_n)$ with $m \neq n$.

First we need to introduce a special family of graphs. Let k and n be positive integers, k odd ($k \geq 3$) and $n \geq 4$, and let C be a cycle on $k[\frac{1}{2}n]$ vertices numbered consecutively. Take $v \notin V(C)$ and join v to every vertex of C not divisible by $[\frac{1}{2}n]$. Call this graph $H(k, n, v)$. Note that $H(k, n, v)$ could be referred to as a skip wheel.

For convenience we introduce the symbol $P_n(v)$. This symbol will denote a path on n vertices with end vertex v.

We state the following two theorems without proof.

Theorem 12. *Let n and k be positive integers, $n \geq 5, k \geq 3$, with k odd.*

(1) *For each coloring of $H(k, n, v)$ either $H(k, n, v)$ contains a monochromatic copy of P_n or a monochromatic copy of $P_3(v)$.*

(2) *There exists a coloring of $H(k, n, v)$ such that it contains a monochromatic copy of $P_3(v)$, but no monochromatic copy of $P_4(v)$ and no monochromatic copy of P_n.*

(3) *For each edge e of $H(k, n, v)$ there exists a coloring of $H(k, n, v) - e$ such that it contains no monochromatic P_n and no monochromatic $P_3(v)$.*

Theorem 13. *Let n be a positive integer, $n \geq 5$. There exists a graph G with distinguished vertex v such that both of the following hold.*

(1) *For each coloring of G either G contains a monochromatic copy of P_n or a monochromatic copy of $P_{n-2}(v)$.*

(2) *There exists a coloring of G such that it contains no monochromatic copy of P_n and no monochromatic copy of $P_{n-1}(v)$.*

Although the proof of this theorem will not be given, we do describe the graphs needed in its proof. For n even let $G = K_{n+n/2-2}$ and designate v as any vertex of G. For n odd consider the graph $K_{n+(n-3)/2}$ and delete $n-2$ edges incident to a fixed vertex. This graph is G with v any vertex of maximal degree. The proof of Theorem 13 follows closely the ideas of the proof given in [7] to determine the Ramsey number for paths.

We now state the desired theorem about $\mathcal{R}(P_n, P_n)$ giving the construction of an infinite subcollection in its proof.

Theorem 14. *For each $n \geq 5$ the family $\mathcal{R}(P_n, P_n)$ has an infinite subcollection of constructible members.*

Proof. Let G' be an edge-minimal subgraph of the graph G of Theorem 13 such that part (1) of the theorem holds. This is the only step in the proof that is not fully effective; but clearly such a G' can be found for each choice of n, simply by a finite search of subgraphs of G. Take two disjoint copies of G' and a copy of the graph $H(k, n, v)$ of Theorem 12 and attach the three graphs together by identifying the labelled vertices v, leaving the graphs otherwise disjoint. Call this graph $L(k)$.

We show that $L(k) \in \mathcal{R}(P_n, P_n)$ for each odd integer k, $k \geq 3$. To see this first color $L(k)$. By Theorem 13(1), if neither copy of G' in $L(k)$ contains a monochromatic P_n, then each copy of G' contains a monochromatic $P_{n-2}(v)$. Call these paths P' and P''. These two paths share only vertex v, being otherwise vertex-disjoint, so that $P' \cup P''$ is a path with $2n-5$ vertices. Since $n \geq 5$ this means that $P' \cup P''$ contains a monochromatic P_n in $L(k)$ or P' and P'' are each monochromatic paths of opposite colors. By Theorem 12(1), either the copy of $H(k, n, v)$ in $L(k)$ contains a monochromatic P_n or a monochromatic $P_3(v)$. Thus either a monochromatic P_n occurs in one of the copies of G' or $H(k, n, v)$, or the paths P', P'' and $P_3(v)$ collectively give a monochromatic P_n in $L(k)$. Hence we have that $L(k) \rightarrow (P_n, P_n)$. Theorem 12(2), (3) and Theorem 13(2) together with the choice of G' show that $L(k) - e \not\rightarrow (P_n, P_n)$. Thus we have that $L(k) \in \mathcal{R}(P_n, P_n)$ for each odd positive integer k, $k \geq 3$. It is clear that $\{L(k)\}_{k \geq k_0}$ is a distinct family where k_0 is such that $|V(H(k_0, n, v))| > |V(G')|$. This completes the proof of the theorem.

There are obvious questions left unanswered, with the most striking one involving the possible finiteness of $\mathcal{R}(G, H)$ when G or H have connectivity two or less. A summary of what is known was given in the introduction. This general problem is quite difficult; in fact it is probably very difficult to determine in general whether $\mathcal{R}(G, H)$ is finite or infinite when both G of H are star-forests with at least one of these forests having components which are single edge stars. Another interesting problem concerns constructing infinite families of $\mathcal{R}(G, H)$ for specific forests (or trees) G and H, as was done in this paper for $G = H = P_n$.

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