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Some Asymptotic Formulas on Generalized Divisor Functions, II

P. ERDÖS AND A. SÁRKÖZY

*Mathematical Institute,
Hungarian Academy of Sciences, Realtanoda U. 13–15, Budapest, Hungary*

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Let A be an infinite sequence of positive integers $a_1 < a_2 < \cdots$ and put $f_A(x) = \sum_{a \in A, a \leqslant x} (1/a)$, $D_A(x) = \max_{1 \leqslant n \leqslant x} \sum_{a \in A, a/n} 1$. In Part I, it was proved that $\lim_{x \to +\infty} \sup D_A(x)/f_A(x) = +\infty$. In this paper, this theorem is sharpened by estimating $D_A(x)$ in terms of $f_A(x)$. It is shown that $\lim_{x \to +\infty} \sup D_A(x) \exp(-c_1(\log f_A(x))^2) = +\infty$ and that this assertion is not true if c_1 is replaced by a large constant c_2.

1

Throughout this paper, we use the following notation: $c, c_1, c_2, \ldots, X_0, X_1, \ldots$ denote positive absolute constants. We denote the number of elements of the finite set S by $|S|$. We write $e^x = \exp(x)$. We denote the least prime factor of n by $p(n)$, while the greatest prime factor of n is denoted by $P(n)$. We write $p^\alpha \| n$ if $p^\alpha | n$ but $p^{\alpha+1} \nmid n$. $\nu(n)$ denotes the number of the distinct prime factors of n, while the number of all the prime factors of n is denoted by $\omega(n)$ so that

$$\nu(n) = \sum_{p|n} 1 \qquad \text{and} \qquad \omega(n) = \sum_{p^\alpha \| n} \alpha.$$

We write

$$\nu(n, y) = \sum_{\substack{p|n \\ p \leqslant y}} 1, \qquad \omega(n, y) = \sum_{\substack{p^\alpha \| n \\ p \leqslant y}} \alpha,$$

$$\nu^+(n, y) = \sum_{\substack{p|n \\ p > y}} 1, \qquad \omega^+(n, y) = \sum_{\substack{p^\alpha \| n \\ p > y}} \alpha$$

and

$$\nu(n, x, y) = \sum_{\substack{p|n \\ x < p \leqslant y}} 1$$

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(so that $\nu(n, n) = \nu^+(n, 1) = \nu(n)$, $\omega(n, n) = \omega^+(n, 1) = \omega(n)$ and $\nu(n, x, y) = \nu(n, y) - \nu(n, x)$). The divisor function is denoted by $d(n)$:

$$d(n) = \sum_{d \mid n} 1.$$

Let A be a finite or infinite sequence of positive integers $a_1 < a_2 < \cdots$. Then we write

$$N_A(x) = \sum_{\substack{a \in A \\ a \leqslant x}} 1,$$

$$f_A(x) = \sum_{\substack{a \in A \\ a \leqslant x}} \frac{1}{a},$$

$$d_A(n) = \sum_{\substack{a \in A \\ a \mid n}} 1$$

(in other words, $d_A(n)$ denotes the number of divisors among the a_i's) and

$$D_A(x) = \max_{1 \leqslant n \leqslant x} d_A(x).$$

In Part I (see [2]), we proved that for an infinite sequence A, we have

$$\lim_{x \to +\infty} \sup \frac{D_A(x)}{f_A(x)} = +\infty. \tag{1}$$

(In [4], Hall proved independently that (1) holds in the special case $\lim_{x \to +\infty} \sup f_A(x)/\log x > 0$. Note that we have $\sum_{1 \leqslant n \leqslant x} d_A(n) = x f_A(x) + O(x)$.) Furthermore, we proved some other related results in [2]. In particular, we proved that

THEOREM 1. *If*

$$\lim_{x \to +\infty} f_A(x) = +\infty \tag{2}$$

then

$$\lim_{x \to +\infty} \sup D_A(x) \left(\frac{\log x}{\log \log x} \right)^{-1} \geqslant 1. \tag{3}$$

(This theorem will be needed in the proof of Theorem 2 below.)

We conjectured in [2] that (1) could be sharpened in the following way:

$$\lim_{x \to +\infty} \sup D_A(x) \exp(-(1-\varepsilon)(\log f_A(x))^2) = +\infty.$$

Sections 2 and 3 will be devoted to the proof of the following slightly weaker estimate:

THEOREM 2. *Assume that for an infinite sequence A of positive integers* $a_1 < a_2 < \cdots$, (2) *holds. Then for all* $\varepsilon > 0$, *we have*

$$\lim_{x \to +\infty} \sup D_A(x) \exp\left(-\left(\frac{e}{16} - \varepsilon\right)(\log f_A(x))^2\right) = +\infty. \tag{4}$$

Furthermore, we show in Section 4 that Theorem 2 is the best possible except for the constant factor in the exponent (and that our conjecture is *false* in its original form):

THEOREM 3. *For all* $\varepsilon > 0$, *there exists an infinite sequence A of positive integers* $a_1 < a_2 < \cdots$ *such that*

(i) *A has density* 1, *i.e.*,

$$\lim_{x \to +\infty} N_A(x)/x = 1; \tag{5}$$

(ii) *we have*

$$\lim_{x \to +\infty} \sup D_A(x) \exp(-(\tfrac{1}{2} + \varepsilon)(\log f_A(x))^2) = 0. \tag{6}$$

Finally, we sketch the proof of three other related results in Section 5. (In particular, Theorem 5 will show that the factor $e/16 - \varepsilon$ in the exponent in (4) cannot be replaced by $e/8 + \varepsilon$.)

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In order to prove Theorem 1, we need some lemmas.

LEMMA 1. *Assume that for an infinite sequence A of positive integers,*

$$\lim_{x \to +\infty} \sup f_A(x) \exp(-(\log \log x)^{1/2}) > 1 \tag{7}$$

holds, and let ε be a fixed positive number. Then there exist infinitely many positive integers x such that

$$f_A(x) > \exp((\log\log x)^{1/2}) \tag{8}$$

and

$$\frac{\log\log x}{\log\log y}\log f_A(y) < (1+\varepsilon)\log f_A(x) \qquad \textit{for all } y > x. \tag{9}$$

Proof. By (7), there exist infinitely many integers z such that

$$f_A(z) > \exp((\log\log z)^{1/2}). \tag{10}$$

Obviously, it is sufficient to show that for such an integer z, there exists an integer x satisfying $x \geqslant z$, (8) and (9). In order to prove this, assume that if $x \geqslant z$ and (8) holds, then there exists an integer y for which (9) does not hold.

Now we are going to show that our assumption implies that there exist positive integers $(z=)\, x_0 < x_1 < x_2 < \cdots$ such that (8) holds with x_k in place of x and

$$\log f_A(x_k) \geqslant (1+\varepsilon)^k \frac{\log\log x_k}{\log\log x_0}\log f_A(x_0) \qquad \text{for } k=0, 1, 2,\ldots. \tag{11}$$

In fact, by (10), $x_0 = z$ satisfies (8) (with x_0 in place of x) and also (11) holds trivially. Assume now that $x_0 < x_1 < \cdots < x_k$ have been defined so that

$$f_A(x_k) > \exp((\log\log x_k)^{1/2}) \tag{12}$$

and (11) hold. Then by (12) (and $x_k \geqslant x_0 = z$), our assumption yields that there exists an integer y for which $y > x_k$ and

$$\frac{\log\log x_k}{\log\log y}\log f_A(y) \geqslant (1+\varepsilon)\log f_A(x_k).$$

Let $x_{k+1} = y$. Then (with respect to (11) and (12)) we have

$$\begin{aligned} f_A(x_{k+1}) = f_A(y) &\geqslant \exp\left((1+\varepsilon)\log f_A(x_k)\frac{\log\log y}{\log\log x_k}\right) \\ &> \exp\left((1+\varepsilon)(\log\log x_k)^{1/2}\frac{\log\log y}{\log\log x_k}\right) \\ &= \exp\left((1+\varepsilon)\left(\frac{\log\log y}{\log\log x_k}\right)^{1/2}(\log\log y)^{1/2}\right) \\ &> \exp((\log\log y)^{1/2}) = \exp((\log\log x_{k+1})^{1/2}) \end{aligned}$$

and

$$\begin{aligned}\log f_A(x_{k+1}) &= \log f_A(y) \geqslant (1+\varepsilon)\frac{\log\log y}{\log\log x_k}\log f_A(x_k)\\ &\geqslant (1+\varepsilon)\frac{\log\log y}{\log\log x_k}(1+\varepsilon)^k\frac{\log\log x_k}{\log\log x_0}\log f_A(x_0)\\ &= (1+\varepsilon)^{k+1}\frac{\log\log x_{k+1}}{\log\log x_0}\log f_A(x_0)\end{aligned}$$

so that both (11) and (12) hold with $k+1$ in place of k, and this proves the existence of a sequence $x_0 < x_1 < \cdots$ having the desired properties.

But if k is large enough (depending on x_0), then (11) yields that

$$\log f_A(x_k) < 2\log\log x_k. \tag{13}$$

On the other hand, obviously we have

$$\begin{aligned}\log f_A(x_k) &= \log\Bigg(\sum_{\substack{a\in A\\ a\leqslant x_k}}\frac{1}{a}\Bigg) \leqslant \log\Bigg(\sum_{a=1}^{x_k}\frac{1}{a}\Bigg)\\ &< \log(\log x_k + c_1) < 2\log\log x_k. \end{aligned}\tag{14}$$

Inequalities (13) and (14) yield a contradiction which completes the proof of Lemma 1.

LEMMA 2. *There exists an absolute constant c_2 such that if x, y and t are positive numbers satisfying*

$$3 \leqslant y \leqslant x \tag{15}$$

and

$$1 \leqslant t < \log\log x, \tag{16}$$

then

$$\sum_{\substack{n\leqslant x\\ v^+(n,y)\leqslant t}}\frac{1}{n} \leqslant c_2 \log y\left(\frac{e\log\log x}{t}\right)^t t^{1/2}.$$

Proof. If $n \leqslant x$ and $v^+(n,y) = m$ then n can be written in the form

$$n = n_1 p_1^{\alpha_1}\cdots p_m^{\alpha_m}, \tag{17}$$

where $n_1 \leqslant x$, $P(n_1) \leqslant y$, $y < p_i \leqslant x$, $p_i \neq p_j$ for $i \neq j$ and $\alpha_1, \dots, \alpha_m$ are positive integers. Furthermore, if n is fixed and $n_1, p_1, \dots, p_m, \alpha_1, \dots, \alpha_m$ satisfy all these conditions, then also the permutations of the prime powers $p_1^{\alpha_1}, \dots, p_m^{\alpha_m}$ satisfy them; thus n has $m!$ representations of the form (17). Hence, with respect to (15) and (16),

$$\begin{aligned}
\sum_{\substack{n \leqslant x \\ v^+(n,y) \leqslant t}} \frac{1}{n} &\leqslant \sum_{m=0}^{[t]} \frac{1}{m!} \left(\sum_{\substack{n_1 \leqslant x \\ P(n_1) \leqslant y}} \frac{1}{n_1} \right) \left(\sum_{y < p \leqslant x} \sum_{\alpha=1}^{+\infty} \frac{1}{p^\alpha} \right)^m \\
&< \sum_{m=0}^{[t]} \frac{1}{m!} \left(\prod_{p \leqslant y} \frac{1}{1 - 1/p} \right) \left(\sum_{p \leqslant x} \frac{1}{p} + c_3 \right)^m \\
&< \sum_{m=0}^{[t]} \frac{1}{m!} \cdot c_4 \log y \cdot (\log \log x + c_5)^m \\
&< c_4 \log y \sum_{m=0}^{[t]} \frac{(\log \log x)^m}{m!} \left(1 + \frac{c_5}{\log \log x} \right)^m \\
&< c_4 \log y \sum_{m=0}^{[t]} \frac{(\log \log x)^{[t]}}{([t])!} \left(1 + \frac{c_5}{\log \log x} \right)^t \\
&< c_6 \log y t \frac{(\log \log x)^{[t]}}{([t])!}
\end{aligned} \tag{18}$$

since

$$\prod_{p \leqslant y} \frac{1}{1 - 1/p} < c_7 \log y,$$

$$\sum_{p \leqslant x} \frac{1}{p} = \log \log x + O(1)$$

and

$$\sum_{p} \sum_{\alpha \geqslant 2} \frac{1}{p^\alpha} < \sum_{n=1}^{+\infty} \sum_{\alpha=2}^{+\infty} \frac{1}{n^\alpha} < +\infty.$$

By using the Stirling formula, we obtain from (18) that

$$\begin{aligned}
\sum_{\substack{n \leqslant x \\ v^+(n,y) \leqslant t}} \frac{1}{n} &< c_8 \log y t \left(\frac{e \log \log x}{[t]} \right)^{[t]} t^{-1/2} \\
&< c_9 \log y \left(\frac{e \log \log x}{t} \right)^t t^{1/2}
\end{aligned}$$

which completes the proof of Lemma 2.

LEMMA 3. *Let E be an arbitrary nonempty set of prime numbers and let*

$$E(x) = \sum_{\substack{p \in E \\ p \leqslant x}} \frac{1}{p}.$$

Then for all $x \geqslant 1$ and $\alpha \geqslant 1$, the number of the integers n satisfying $1 \leqslant n \leqslant x$ and

$$\sum_{\substack{p \mid n \\ p \in E}} 1 > \alpha E(x)$$

is $\leqslant c_{10} x \exp((\alpha - 1 - \alpha \log \alpha) E(x))$.

This lemma is due to K. K. Norton; see (5.16) and (1.11) in [6], also [7].

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In this section, we complete the proof of Theorem 2.

Let ε be a small but fixed positive number such that $\varepsilon < 1$.

Assume first that

$$\lim_{x \to +\infty} \sup f_A(x) \exp(-(\log \log x)^{1/2}) \leqslant 1.$$

Then for $x > X_0$, we have

$$f_A(x) < 2 \exp((\log \log x)^{1/2}).$$

Hence

$$\begin{aligned} &\exp\left(\left(\frac{1}{4} - \varepsilon\right)(\log f_A(x))^2\right) \\ &\quad < \exp\left(\left(\frac{1}{4} - \varepsilon\right)(\log(2\exp((\log\log x)^{1/2}))^2\right) \\ &\quad < \exp\left(\frac{1}{3}\log\log x\right) < \exp\left(\frac{1}{2}(\log\log x - \log\log\log x)\right) \\ &\quad = \left(\frac{\log x}{\log\log x}\right)^{1/2} = o\left(\frac{\log x}{\log\log x}\right). \end{aligned} \tag{19}$$

By (2), we may apply Theorem 1, and we find that (3) holds. Inequalities (3) and (19) yield (4).

Assume now that

$$\lim_{x\to+\infty} \sup f_A(x) \exp(-(\log\log x)^{1/2}) > 1.$$

Then by Lemma 1 (with $\varepsilon/2$ in place of ε), there exist infinitely many integers x such that

$$f_A(x) > \exp((\log\log x)^{1/2}) \tag{20}$$

and

$$\frac{\log\log x}{\log\log y} \log f_A(y) < \left(1+\frac{\varepsilon}{2}\right) \log f_A(x) \qquad \text{for all } x<y. \tag{21}$$

Obviously, in order to prove that (4) holds also in this case, it is sufficient to show that if $x > X_1(\varepsilon)$ and x satisfies (20) and (21), then there exists an integer u satisfying

$$x \leqslant u \leqslant \exp((\log x)^2) \tag{22}$$

and

$$D_A(u) > \exp\left(\left(\frac{e}{16}-\frac{\varepsilon}{2}\right)(\log f_A(u))^2\right). \tag{23}$$

Assume that x is large (in terms of ε) and x satisfies both (20) and (21). Let us write

$$y = \exp\{(\log f_A(x))^3\} \tag{24}$$

and

$$g(u) = \left(\frac{e\log\log x}{u}\right)^u, \qquad h(u) = \frac{u}{(\log u)^4} \qquad (\text{for } u>0).$$

Obviously, we have

$$f_A(x) = \sum_{\substack{a\in A\\ a\leqslant x}} \frac{1}{a} \leqslant \sum_{a\leqslant x} \frac{1}{a} < \log x + c_{11}. \tag{25}$$

It can be shown easily that the function $h(u)$ is increasing for $u > U_0$. Thus, by (20) and (25), we have

$$\begin{aligned} h(f_A(x)) &> h(\exp((\log\log x)^{1/2})) \\ &= \frac{\exp((\log\log x)^{1/2})}{(\log\log x)^2} \, (>(\log\log x)^2) \end{aligned} \tag{26}$$

and

$$h(f_A(x)) < h(\log x + c_{11}) = \frac{\log x + c_{11}}{(\log(\log x + c_{11}))^4} < \log x \tag{27}$$

for sufficiently large x. Furthermore, for $1 \leqslant u < \log\log x$, the function $g(u)$ is continuous and increasing since

$$g'(u) = (\log\log\log x - \log u)\left(\frac{e\log\log x}{u}\right)^u > 0,$$

and by (26) and (27), we have

$$g(1) = e\log\log x < h(f_A(x))$$

and

$$g(\log\log x) = \log x > h(f_A(x)).$$

Thus, there exists a uniquely determined real number t such that

$$1 < t < \log\log x \tag{28}$$

and

$$g(t) = h(f_A(x)). \tag{29}$$

We need lower and upper bounds for this number t. By (28), we have

$$\frac{f_A(x)}{(\log f_A(x))^4} = h(f_A(x)) = g(t) = \left(\frac{e\log\log x}{t}\right)^t > e^t;$$

hence

$$t < \log\frac{f_A(x)}{(\log f_A(x))^4} < \log f_A(x). \tag{30}$$

On the other hand, by (29), we have

$$\begin{aligned}\frac{t\log\log x}{(\log h(f_A(x)))^2} &= \frac{t\log\log x}{(\log g(t))^2} = \frac{t\log\log x}{t^2(1+\log\log\log x - \log t)^2}\\ &= \frac{1}{\dfrac{t}{\log\log x}\left(1-\log\dfrac{t}{\log\log x}\right)^2}\\ &= \frac{1}{v(1-\log v)^2}\end{aligned} \tag{31}$$

where (with respect to (28))

$$0 < v = \frac{t}{\log \log x} < 1.$$

But a simple computation shows that for $0 < \xi < 1$, the function

$$\varphi(\xi) = \frac{1}{\xi(1 - \log \xi)}$$

assumes its minimal value at $\xi = 1/e$ so that

$$\varphi(v) = \frac{1}{v(1 - \log v)^2} \geqslant \varphi\left(\frac{1}{e}\right) = \frac{e}{4}. \tag{32}$$

By (31) and (32) we have

$$\begin{aligned} t &= \frac{1}{v(1 - \log v)^2} \cdot \frac{(\log h(f_A(x)))^2}{\log \log x} \\ &\geqslant \frac{e}{4} \frac{(\log(f_A(x)/(\log f_A(x))^4))^2}{\log \log x} = \left(1 - \frac{\varepsilon}{4}\right) \frac{e}{4} \frac{(\log f_A(x))^2}{\log \log x} \end{aligned} \tag{33}$$

if x is sufficiently large.

Let A^* denote the set of the integers a such that $a \leqslant x$, $a \in A$ and

$$v^+(a, y) > t.$$

By (24) and (25), we have

$$\begin{aligned} (3\leqslant) \qquad y &= \exp((\log f_A(x))^3) \\ &< \exp((\log(\log x + c_{10}))^3) \qquad (\leqslant x). \end{aligned} \tag{34}$$

By (28) and (34), both (15) and (16) hold; thus Lemma 2 can be used in order to estimate $f_{A^*}(x)$, and we obtain that

$$\begin{aligned} f_{A^*}(x) &= \sum_{a \in A^*} \frac{1}{a} = \sum_{\substack{a \leqslant x, a \in A \\ v^+(a,y) > t}} \frac{1}{a} = \sum_{a \leqslant x, a \in A} \frac{1}{a} - \sum_{\substack{a \leqslant x, a \in A \\ v^+(a,y) \leqslant t}} \frac{1}{a} \\ &= f_A(x) - \sum_{\substack{a \leqslant x, a \in A \\ v^+(a,y) \leqslant t}} \frac{1}{a} \geqslant f_A(x) - \sum_{\substack{a \leqslant x \\ v^+(a,y) \leqslant t}} \frac{1}{a} \\ &\geqslant f_A(x) - c_2 \log y \left(\frac{e \log \log x}{t}\right)^t t^{1/2}. \end{aligned}$$

Hence with respect to (24), (29) and (30)

$$f_{A^*}(x) \geqslant f_A(x) - c_2(\log f_A(x))^3 \frac{f_A(x)}{(\log f_A(x))^4} (\log f_A(x))^{1/2}$$

$$= f_A(x) \left(1 - \frac{c_2}{(\log f_A(x))^{1/2}}\right) > \frac{1}{2} f_A(x). \tag{35}$$

Let us write

$$k = [\log x]$$

and let S denote the set of the integers n such that

$$n \leqslant x^k$$

and n can be represented in the form

$$a_{i_1} a_{i_2} \cdots a_{i_k} m = n, \tag{36}$$

where $a_{i_1} \in A^*$, $a_{i_2} \in A^*, \ldots, a_{i_k} \in A^*$ (and m is positive integer). For fixed $n \in S$, let $g(n)$ denote the number of representations of n in the form (36).

Then by (35), we have

$$\sum_{n \in S} g(n) = \sum_{n \in S} \left(\sum_{\substack{a_{i_1} \in A^*, \ldots, a_{i_k} \in A^* \\ a_{i_1} \cdots a_{i_k} / n}} 1 \right)$$

$$= \sum_{a_{i_1} \in A^*, \ldots, a_{i_k} \in A^*} \left(\sum_{\substack{n \leqslant x^k \\ a_{i_1} \cdots a_{i_k} / n}} 1 \right)$$

$$= \sum_{a_{i_1} \in A^*, \ldots, a_{i_k} \in A^*} \left[\frac{x^k}{a_{i_1} \cdots a_{i_k}} \right]$$

$$> \sum_{a_{i_1} \in A^*, \ldots, a_{i_k} \in A^*} \frac{1}{2} \frac{x^k}{a_{i_1} \cdots a_{i_k}}$$

$$= \frac{1}{2} x^k \sum_{a_{i_1} \in A^*, \ldots, a_{i_k} \in A^*} \frac{1}{a_{i_1} \cdots a_{i_k}}$$

$$= \frac{1}{2} x^k f_{A^*}^k(x) > \frac{1}{2} x^k \left(\frac{1}{2} f_A(x)\right)^k \tag{37}$$

since $a \in A^*$ implies that $a \leqslant x$, and $[u] > \frac{1}{2}u$ for all $u \geqslant 1$.

On the other hand, we have

$$\sum_{n\in S} g(n)=\sum_{n\in S}\left(\sum_{\substack{a_{i_1}\in A^*,\dots,a_{i_k}\in A^*\\ a_{i_1}\cdots a_{i_k}/n}} 1\right)$$

$$\leqslant \sum_{n\in S}\left(\sum_{\substack{a_{i_1}\in A^*\\ a_{i_1}/n}} 1\right)\cdots\left(\sum_{\substack{a_{i_k}\in A^*\\ a_{i_k}/n}} 1\right)=\sum_{n\in S}(d_{A^*}(n))^k$$

$$\leqslant \sum_{n\in S}(d_A(n))^k \leqslant \sum_{n\in S}(D_A(x^k))^k=(D_A(x^k))^k\,|S|. \tag{38}$$

Let S_1 denote the set of the integers n such that $n\in S$ and

$$\omega^+(n,y)-v^+(n,y)>\frac{\varepsilon}{6}kt \tag{39}$$

and write

$$S_2=S-S_1.$$

Then we have $S=S_1\cup S_2$ so that

$$|S|\leqslant |S_1|+|S_2|. \tag{40}$$

First we estimate $|S_1|$. Let $n\in S_1$, $n=(p_1^{\alpha_1}\cdots p_r^{\alpha_r})^2 n_1$ where $y<p_1<p_2<\cdots<p_r$, and $p>y$ implies that $p^2\nmid n_1$. Then obviously,

$$\omega^+(n,y)-v^+(n,y)\leqslant 2(\alpha_1+\alpha_2+\cdots+\alpha_r). \tag{41}$$

Inequalities (39) and (41) yield that

$$p_1^{\alpha_1}\cdots p_r^{\alpha_r}>y^{\alpha_1}\cdots y^{\alpha_r}=y^{\alpha_1+\cdots+\alpha_r}\geqslant y^{(\omega^+(n,y)-v^+(n,y))/2}>y^{(\epsilon/12)kt}.$$

Thus, writing $j=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, we obtain that for $n\in S_1$, there exists a positive integer j such that j^2/n and $j>y^{(\epsilon/12)kt}$. Hence with respect to (24),

$$|S_1|\leqslant \sum_{j>y^{(\epsilon/12)kt}}\ \sum_{\substack{n\leqslant x^k\\ j^2/n}} 1$$

$$=\sum_{j>y^{(\epsilon/12)kt}}\left[\frac{x^k}{j^2}\right]<x^k\sum_{j>y^{(\epsilon/12)kt}}\frac{1}{j^2}<x^k\sum_{j>y^{(\epsilon/12)kt}}\frac{1}{(j-1)j}$$

$$=x^k\sum_{j>y^{(\epsilon/12)kt}}\left(\frac{1}{j-1}-\frac{1}{j}\right)=x^k\frac{1}{[y^{(\epsilon/12)kt}]}<x^k\frac{1}{y^{(\epsilon/13)kt}}$$

$$= x^k \exp\left(-\frac{\varepsilon}{13} kt \log y\right)$$

$$= x^k \exp\left(-\frac{\varepsilon}{13} kt (\log f_A(x))^3\right) < x^k \exp(-k(\log f_A(x))^{5/2}) \tag{42}$$

for sufficiently large x.

Now we estimate $|S_2|$. If $n \in S_2$ then $n \notin S_1$; thus we have

$$\omega^+(n, y) - v^+(n, y) \leqslant \frac{\varepsilon}{6} kt. \tag{43}$$

Furthermore, $n \in S_2$ implies that $n \in S$ and thus n can be represented in the form (36). Hence with respect to (43), for $n \in S_2$ we have

$$\begin{aligned} v(n, y, x) &= \omega(n, y, x) - (\omega(n, y, x) - v(n, y, x)) \\ &\geqslant \omega(n, y, x) - (\omega^+(n, y) - v^+(n, y)) \geqslant \omega(n, y, x) - \frac{\varepsilon}{6} kt \\ &= \omega(a_{i_1} \cdots a_{i_k} m, y, x) - \frac{\varepsilon}{6} kt \\ &= \sum_{j=1}^{k} \omega(a_{i_j}, y, x) + \omega(m, y, x) - \frac{\varepsilon}{6} kt \\ &= \sum_{j=1}^{k} \omega^+(a_{i_j}, y) + \omega(m, y, x) - \frac{\varepsilon}{6} kt \\ &\geqslant \sum_{j=1}^{k} v^+(a_{i_j}, y) - \frac{\varepsilon}{6} kt > \sum_{j=1}^{k} t - \frac{\varepsilon}{6} kt \\ &= \left(1 - \frac{\varepsilon}{6}\right) kt \end{aligned}$$

so that

$$|S_2| \leqslant \sum_{\substack{n \leqslant x^k \\ v(n,y,x) > (1-\varepsilon/6)kt}} 1. \tag{44}$$

Let E denote the set of the prime numbers such that $y < p \leqslant x$. Then

$$\sum_{\substack{p \mid n \\ p \in E}} 1 = v(n, y, x)$$

and we have

$$E(x^k) = E(x) = \sum_{p \in E} \frac{1}{p} = \sum_{p \leqslant x} \frac{1}{p} - \sum_{p \leqslant y} \frac{1}{p}$$

$$= \log \log x - \log \log y + O\left(\frac{1}{\log x}\right) < \log \log x \tag{45}$$

(for large enough x) since

$$\sum_{p \leqslant u} \frac{1}{p} = \log \log u + c + O\left(\frac{1}{\log u}\right).$$

Write $\alpha = (1 - \varepsilon/6)\, kt/E(x)$. Then for large x, $\alpha \geqslant 1$ holds trivially (by (45)). Thus, by Lemma 3, we obtain (with respect to (28), (33), (44) and (45)) that for large x,

$$|S_2| \leqslant \sum_{\substack{n \leqslant x^k \\ \nu(n,y,x) > (1-\varepsilon/6)kt}} 1$$

$$= \sum_{\substack{n \leqslant x^k \\ \nu(n,y,x) > \alpha E(x^k)}} 1 < c_{10} x^k \exp((\alpha - 1 - \alpha \log \alpha)\, E(x^k))$$

$$= c_{10} x^k \exp\left(\left(1 - \frac{\varepsilon}{6}\right) kt - E(x) - \left(1 - \frac{\varepsilon}{6}\right) kt \log \frac{(1-\varepsilon/6)\, kt}{E(x)}\right)$$

$$< c_{10} x^k \exp\left(\left(1 - \frac{\varepsilon}{6}\right) kt \left(1 - \log \frac{(1-\varepsilon/6)\, kt}{E(x)}\right)\right)$$

$$< x^k \exp\left(-\left(1 - \frac{\varepsilon}{5}\right) kt \log k\right)$$

$$< x^k \exp\left(-\left(1 - \frac{\varepsilon}{4}\right) k \left(1 - \frac{\varepsilon}{4}\right) \frac{e}{4} \frac{(\log f_A(x))^2}{\log \log x} \log \log x\right)$$

$$< x^k \exp\left(-\left(1 - \frac{\varepsilon}{2}\right) \frac{e}{4} k (\log f_A(x))^2\right). \tag{46}$$

Inequalities (40), (42) and (46) yield that

$$|S| \leqslant |S_1| + |S_2|$$

$$\leqslant x^k \left(\exp(-k(\log f_A(x))^{5/2}) + \exp\left(-\left(1 - \frac{\varepsilon}{2}\right) \frac{e}{4} k (\log f_A(x))^2\right)\right)$$

$$< x^k \exp\left(-(1-\varepsilon) \frac{e}{4} k (\log f_A(x))^2\right). \tag{47}$$

By (37, (38) and (47), we have

$$\frac{1}{2}x^k\left(\frac{1}{2}f_A(x)\right)^k < (D_A(x^k))^k\,|S|$$

$$< (D_A(x^k))^k\,x^k \exp\left(-(1-\varepsilon)\frac{e}{4}k(\log f_A(x))^2\right).$$

Thus, writing $u = x^k$, we obtain, in view of (21) that

$$D_A(u) = D_A(x^k) > \frac{1}{2}\cdot\frac{1}{2}f_A(x)\exp\left((1-\varepsilon)\frac{e}{4}(\log f_A(x))^2\right)$$

$$> \exp\left((1-\varepsilon)\frac{e}{4}(\log f_A(x))^2\right)$$

$$> \exp\left((1-\varepsilon)\frac{e}{4}\left(\frac{1}{1+\varepsilon/2}\right)^2\left(\frac{\log\log x}{\log\log y}\log f_A(y)\right)^2\right)$$

$$> \exp\left((1-\varepsilon)\frac{e}{4}\left(1-\frac{\varepsilon}{2}\right)^2\left(\frac{\log\log x}{\log\log(x^{\log x})}\log f_A(u)\right)^2\right)$$

$$> \exp\left((1-\varepsilon)^2\frac{e}{4}\left(\frac{1}{2}\log f_A(u)\right)^2\right)$$

$$> \exp\left((1-2\varepsilon)\frac{e}{16}(\log f_A(u))^2\right)$$

$$> \exp\left(\left(\frac{e}{16}-\frac{\varepsilon}{2}\right)(\log f_A(u))^2\right)$$

and

$$x \leqslant u = x^{[\log x]} \leqslant x^{\log x} = \exp((\log x)^2)$$

so that both (22) and (23) hold and this completes the proof of Theorem 2.

4

In order to prove Theorem 3, we need the following lemma:

LEMMA 4. *Let $F(x) \to +\infty$ and δ be a fixed positive number. Let A denote the sequence of positive integers n such that*

$$\text{(i)}\quad |\nu(n,y) - \log\log y| < \delta\log\log y \qquad \textit{for all } F(n) < y \leqslant n. \qquad (48)$$

Then A_δ gas density 1.

This lemma can be proved by the methods of probabilistic number theory (see [1, 5]).

Using the same notations as in Lemma 4, let A_δ^* denote the set of the integers n such that $n \in A_\delta$ and

(ii)

$$\text{if} \quad j > F(n) \qquad \text{then} \quad j^2 \nmid n \tag{49}$$

(in other words, A_δ^* denotes the set of the integers n satisfying both (i) and (ii)). Obviously, (ii) holds for all but $o(x)$ integers n; thus by Lemma 4, also A_δ^* has density 1.

Now we are going to show that choosing $F(x) = \log\log\log x$ and $\delta = \varepsilon/100$ in the definition of this sequence A_δ^*, we obtain a sequence $A = A_\delta^*$ which satisfies conditions (i) and (ii) in Theorem 3.

In fact, (i) holds since A_δ^* has density 1 (by Lemma 4). In order to show that also (ii) holds, let n denote an arbitrary integer, and assume that d/n and $d \in A_\delta^*$. Let $k = [4/\varepsilon] + 1$ and write

$$n = n_0 n_1 n_2 \cdots n_k, \qquad d = d_0 d_1 d_2 \cdots d_k,$$

where

$$P(n_0) \leqslant F(n), \tag{50}$$

$$F(n) < p(n_1) \leqslant P(n_1) \leqslant \exp((\log n)^{1/k}), \tag{51}$$

$$\exp((\log n)^{(i-1)/k}) < p(n_i) \leqslant P(n_i) \leqslant \exp((\log n)^{i/k}) \qquad \text{for } i = 2, 3, \ldots, k \tag{52}$$

and

$$d_i \mid n_i \qquad \text{for } i = 1, 2, \ldots, k. \tag{53}$$

By (50) and (53), d_0 may assume at most

$$\begin{aligned} d(n_0) &= \sum_{p^\alpha \| n_0} (\alpha + 1) \leqslant \sum_{p^\alpha \| n_0} \left(\frac{\log n_0}{\log 2} + 1\right) \\ &= \left(\frac{\log n_0}{\log 2} + 1\right)^{v(n_0)} \leqslant (2 \log n_0)^{\pi(F(n))} \leqslant (2 \log n)^{F(n)} \\ &= (2 \log n)^{\log\log\log n} = \exp(2 \log\log n \log\log\log n) \end{aligned}$$

distinct values for large n.

Furthermore, by (49) and (51), $d_1 \mid n_1$ implies that

$$d_1 \,\Big|\, \prod_{p \mid n_1} p;$$

thus, the prime factors of d_1 can be chosen from the

$$\nu(n_1) \leqslant \log n_1 \leqslant \log n$$

prime factors of n_1, and by (48) and (51), their number is at most

$$\begin{aligned}\nu(d_1) &\leqslant \nu(d, \exp((\log n)^{1/k})) \leqslant (1+\delta) \log\log(\exp(\log n)^{1/k}))\\ &= (1+\delta)\frac{1}{k}\log\log n.\end{aligned}$$

Thus, d_1 may assume at most

$$\begin{aligned}&\sum_{0 \leqslant i \leqslant (1+\delta)(1/k)\log\log n} \binom{\nu(n_1)}{i}\\ &\leqslant \sum_{0 \leqslant i \leqslant (1+\delta)(1/k)\log\log n} (\nu(n_1))^i \leqslant \sum_{0 \leqslant i \leqslant (1+\delta)(1/k)\log\log n} (\log n)^i\\ &\leqslant \log\log n (\log n)^{(1+\delta)(1/k)\log\log n}\\ &= \exp\left(\log\log\log n + \left(1 + \frac{\varepsilon^2}{100}\right)\frac{1}{k}(\log\log n)^2\right)\\ &< \exp\left(\left(1 + \frac{\varepsilon^2}{99}\right)\frac{1}{k}(\log\log n)^2\right)\end{aligned}$$

distinct values.

Finally, (49), (52) and (53) imply that for $i = 2, 3, \ldots, k$, we have

$$d_i \;\Big|\; \prod_{p \mid n_i} p;$$

thus, the prime factors of d_i can be selected from the $\nu(n_i)$ prime factors of n_i. By (52), we have

$$\begin{aligned}n \geqslant n_i \geqslant \prod_{p \mid n_i} p \geqslant \prod_{p \mid n_i} p(n_i) \geqslant \prod_{p \mid n_i} \exp((\log n)^{(i-1)/k})\\ = \exp(\nu(n_i)(\log n)^{(i-1)/k});\end{aligned}$$

hence

$$\nu(n_i) \leqslant (\log n)/(\log n)^{(i-1)/k} = (\log n)^{1-(i-1)/k}.$$

Furthermore, by (48), (52) and (53),

$$\begin{aligned} v(d_i) &= v(d, \exp((\log n)^{i/k})) - v(d, \exp((\log n)^{(i-1)/k})) \\ &< (1+\delta) \log\log(\exp((\log n)^{i/k})) - (1-\delta) \log\log(\exp((\log n)^{(i-1)/k})) \\ &= \left((1+\delta)\frac{i}{k} - (1-\delta)\frac{i-1}{k}\right) \log\log n \\ &= \left(\frac{1}{k} + \delta\frac{2i-1}{k}\right) \log\log n < \left(\frac{1}{k} + 2\delta\right) \log\log n. \end{aligned}$$

Thus, d_i may assume at most

$$\begin{aligned} &\sum_{0 \leqslant j \leqslant (1/k+2\delta)\log\log n} \binom{v(n_i)}{j} \\ &\qquad \leqslant \sum_{0 \leqslant j \leqslant (1/k+2\delta)\log\log n} (v(n_i))^j \\ &\leqslant \log\log n((\log n)^{1-(i-1)/k}) \left(\frac{1}{k} + 2\delta\right) \log\log n \\ &\qquad < \exp\left(\log\log\log n + \left(\left(\frac{1}{k} - \frac{i-1}{k^2}\right) + 2\delta\right)(\log\log n)^2\right) \\ &\qquad < \exp\left(\left(\left(\frac{1}{k} - \frac{i-1}{k^2}\right) + \frac{\varepsilon^2}{49}\right)(\log\log n)^2\right) \end{aligned}$$

values.

Summarizing our estimates above, we obtain that the product of the d_i's, i.e., d can be chosen in at most

$$\begin{aligned} d_A(n) &< \exp(2\log\log n \log\log\log n) \cdot \exp\left(\left(1 + \frac{\varepsilon^2}{99}\right)\frac{1}{k}(\log\log n)^2\right) \\ &\quad \cdot \prod_{i=2}^{k} \exp\left(\left(\left(\frac{1}{k} - \frac{i-1}{k^2}\right) + \frac{\varepsilon^2}{49}\right)(\log\log n)^2\right) \\ &< \exp\left(\left(\frac{\varepsilon^2}{99} + \frac{1}{k} + \frac{\varepsilon^2}{99k} + \sum_{i=2}^{k}\left(\frac{1}{k} - \frac{i-1}{k^2}\right) + k\frac{\varepsilon^2}{49}\right) \cdot (\log\log n)^2\right) \\ &< \exp\left(\left(1 - \frac{(k-1)k/2}{k^2} + \left(\frac{1}{99} + \frac{1}{99} + \frac{k}{49}\right)\varepsilon^2\right)(\log\log n)^2\right) \\ &< \exp\left(\left(\frac{1}{2} + \frac{1}{2k} + (k+2)\frac{\varepsilon^2}{49}\right)(\log\log n)^2\right) \end{aligned}$$

$$= \exp\left(\left(\frac{1}{2} + \frac{1}{2([4/\varepsilon]+1)} + \left(\left[\frac{4}{\varepsilon}\right]+3\right)\frac{\varepsilon^2}{49}\right)(\log\log n)^2\right)$$

$$< \exp\left(\left(\frac{1}{2} + \frac{\varepsilon}{8} + 2\cdot\frac{4}{\varepsilon}\cdot\frac{\varepsilon^2}{49}\right)(\log\log n)^2\right)$$

$$< \exp\left(\left(\frac{1}{2} + \frac{\varepsilon}{2}\right)(\log\log n)^2\right) \tag{54}$$

ways.

Furthermore, $A = A_\delta^*$ has density 1; thus, for large x we have

$$f_A(x) > \frac{1}{2}\sum_{i \leqslant x}\frac{1}{i} > \frac{1}{3}\log x;$$

hence

$$\log f_A(x) > \log\log x - 2. \tag{55}$$

Inequalities (54 and (55) yield (6) and this completes the proof of Theorem 3.

5

In this section, we formulate three results which can be proved by the same methods as Theorems 2 and 3, respectively.

THEOREM 4. *For all $\varepsilon > 0$, there exists a number $X_0 = X_0(\varepsilon)$ such that if $x > X_0$ and A is a sequence of positive integers satisfying*

$$N_A(x) > \frac{x}{\log x}\exp((\log\log x)^{1/2}),$$

then there exists an integer u such that

$$x \leqslant u \leqslant \exp((\log x)^2) \tag{56}$$

and

$$d_A(u) > \frac{N_A(x)}{x}\exp\left(\left(\frac{e}{4} - \varepsilon\right)\left(\log\frac{N_A(x)\log x}{x}\right)^2\right) \tag{57}$$

(*so that for $\alpha > 0$, $x > X_1(\alpha, \varepsilon)$, $N_A(x) > \alpha x$* we have

$$d_A(u) > \exp\left(\left(\frac{e}{4} - \varepsilon\right)(\log\log x)^2\right)\bigg).$$

Note that for "small" values of $N_A(x)$, the following trivial inequality can be used in order to estimate $D_A(\exp((\log x)^2))$: if $N_A(x) > \log x$, then we have

$$D_A(\exp((\log x)^2)) \geqslant d_A(a_1 a_2 \cdots a_{[\log x]}) \geqslant [\log x].$$

Theorem 4 can be proved in the same way as Theorem 2. However, Lemma 2 must be replaced by an upper estimate for $\sum_{n \leqslant x, v^+(n,y) \leqslant t} 1$:

LEMMA 5. *There exist absolute constants c_{12} and c_{13} such that if x, y and t are positive real numbers satisfying*

$$3 \leqslant y < x^{c_{12}}$$

and

$$1 \leqslant t < \log\log x - \log\log y$$

then we have

$$\sum_{\substack{n \leqslant x \\ v^+(n,y) \leqslant t}} 1 < c_{13} \frac{x}{\log x} \log y \left(\frac{e \log\log x}{t}\right)^t t^{1/2}.$$

This lemma is a consequence of a theorem of Halász; see [3], see also [6, pp. 687–689].

THEOREM 5. *For all $\varepsilon > 0$, there exists an infinite sequence A of positive integers $a_1 < a_2 < \cdots$ such that*

$$\text{(i)} \qquad \lim_{x \to +\infty} \inf \frac{N_A(x)}{x} (\log x)^{1-2/e+\varepsilon} = +\infty, \tag{58}$$

$$\text{(ii)} \qquad \lim_{x \to +\infty} \sup D_A(x) \exp\left(-\left(\frac{e}{8} + \varepsilon\right)(\log f_A(x))^2\right) = 0. \tag{59}$$

(Thus the factor $e/16 - \varepsilon$ in the exponent in (4) cannot be replaced by $e/8 + \varepsilon$.)

Sketch of the Proof. Let B_δ denote the sequence consisting of the positive integers n such that

(i) $|v(n,y) - (1/e)\log\log y| < \delta \log\log y$ for all $\log\log\log n < y \leqslant n$;

(ii) if $j > \log\log\log n$, then $j^2 \nmid n$.

By using the results of Halász and Norton (see [3, 6]), it can be shown that if δ is sufficiently small in terms of ε, then for $x > X_0(\varepsilon)$; the sequence $A = B_\delta$ satisfies

$$N_A(x) > \frac{x}{\log x} (\log x)^{2/e - \varepsilon/10} \qquad (\text{for } x > X_0(\varepsilon)); \tag{60}$$

and this implies (58).

On the other hand, it can be proved by the method used in the proof of Theorem 3 that if δ is sufficiently small in terms of ε, then for $x > X_1(\varepsilon)$, the sequence $A = B_\delta$ satisfies also

$$d_A(x) < \exp\left(\left(\frac{1}{2e} + \frac{\varepsilon}{10}\right)(\log\log x)^2\right). \tag{61}$$

(60) and (61) yield (59).

THEOREM 6. *If $\varepsilon > 0$ and $x > X_0(\varepsilon)$, then there exists a sequence A of positive integers $a_1 < a_2 < \cdots$ such that $A \subset \{1, 2, \ldots, x\}$,*

(i) $N_A(x) > x(\log x)^{-1+2/e-\varepsilon}$,

(ii) $d_A(u) < \dfrac{N_A(x)}{x} \exp\left(\left(\dfrac{3e}{8} + \varepsilon\right)\left(\log \dfrac{N_A(x)\log x}{x}\right)^2\right)$

for all u satisfying (56).

(This theorem shows that in (57) in Theorem 4, the factor $e/4 - \varepsilon$ cannot be replaced by $3e/8 + \varepsilon$.)

In order to prove Theorem 6, put $A = B_\delta \cap [0, x]$ where B_δ is defined in the proof of Theorem 5. By (60) and by using the same method as in the proof of Theorem 3, it can be shown that if δ is sufficiently small and x is sufficiently large in terms of ε then both (i) and (ii) hold.

REFERENCES

1. P. ERDŐS, On the distribution function of additive functions, *Ann. Math.* **47** (1946), 1–20.
2. P. ERDŐS AND A. SÁRKÖZY, Some asymptotic formulas on generalized divisor functions, I, "Studies in Pure Mathematics to the Memory of Paul Turán," Akadémiai Kiadó, Budapest, in press.
3. G. HALÁSZ, Remarks to „On the distribution of additive and the mean values of multiplicative arithmetic functions", *Acta Math. Acad. Sci. Hungar.* **23** (1972), 425–432.

4. R. R. HALL, On a conjecture of Erdös and Sárközi, *Bull. London Matth. Soc.* **12** (1980), 21–24.
5. J. KUBILIUS, "Probabilistic Methods in the Theory of Numbers," Translation of Math. Monographs, Vol.II, Amer. Math. Soc. Providence, R. I. 1964.
6. K. K. NORTON, On the number of restricted prime factors of an integer, I. *Illinois J. Math.* **20** (1976), 681–705.
7. K. K. NORTON, On the number of restricted prime factors of an integer, III, to appear.

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