The Ramsey Number for the Pair Complete Bipartite Graph-Graph of Limited Degree

by

S.A. Burr
City College, C.U.N.Y.
New York, N.Y.

P. Erdös
Hungarian Academy of Sciences
Budapest, Hungary

R.J. Faudree, C.C. Rousseau
and R.H. Schelp
Memphis State University
Memphis, Tennessee

ABSTRACT

A connected graph G is said to be F-good if the Ramsey number $r(F,G)$ has the value $(\chi(F)-1)(p(G)-1) + s(F)$, where $s(F)$ is the minimum number of vertices in some color class under all vertex colorings by $\chi(F)$ colors. It has been previously shown that certain "large" order graphs G with "few" edges are F-good when F is a fixed multipartite graph. We show when F is a complete bipartite graph that this edge condition can be relaxed.

Let F and G be (simple) graphs. The Ramsey number $r(F,G)$ is the smallest positive integer p such that if each edge of the complete graph is colored one of the two colors red or blue, then either the red subgraph contains a copy of F or the blue subgraph contains a copy of G. Two surveys on this subject have been written by S. A. Burr [2,3]. Notation throughout the paper follows that given in [1].

Calculation of the Ramsey number for an arbitrary pair of graphs is known to be an extremely difficult problem. For a given pair of graphs, a starting place is knowing which graphical parameters affect the value of the Ramsey number.

Consider the pair (K_m, T_n) where T_n denotes a tree on n vertices and K_m (as usual) the complete graph on m vertices. V. Chvátal first observed that $r(K_m, T_n) = (m-1)(n-1) + 1$ [11]. The canonical example which determines the lower bound for this number is a two-colored $K_{(m-1)(n-1)}$ with the blue subgraph consisting of $m-1$ disjoint copies of K_{n-1} and with the red subgraph as its complementary graph. The example indicates that the important parameters for this pair of graphs are the chromatic number of K_m and the order of the connected graph T_n.

The lower bound implicit in Chvátal's result can be generalized. If F is a graph, let $s(F)$ (the chromatic surplus of F) denote the smallest number of vertices in a color class under any χ coloring of F. The symbols s or $s(F)$ will always denote this quantity.

Lemma 1 [4].

If F and G are graphs, with $p(G) \geq s(F)$, and where G is connected, then

$$r(F,G) \geq (\chi(F)-1)(p(G)-1) + s(F).$$

With this lemma in mind, we say that a connected graph G is <u>F-good</u> if $r(F,G) = (\chi(F)-1)(p(G)-1) + s(F)$, that is, if Lemma 1 is sharp.

One would expect G to be F-good if G were "almost" a tree and F were "almost" a complete graph. Some families of graphs known to be K_m-good or F-good where F is almost a complete graph are given in [4,5,6,7,13].

Recently the following results have been obtained which show that large order trees T_n are F-good for certain fixed graphs F.

<u>Theorem.</u> [8]

$$r(K_\ell + \overline{K}_m, T_n) = \ell(n-1) + 1 \quad \text{for} \quad \ell \geq 2 \quad \text{and} \quad 3m \leq m$$

<u>Theorem.</u> [8]

Let $\ell_1 \leq \ell_2 \ldots \leq \ell_m$ be fixed positive integers. For n sufficiently large

$$r(K(1,1,\ell_1,\ell_2,\ldots,\ell_m),T_n) = (m+1)(n-1) + 1.$$

In each of the last two theorems a bit more is true. The same Ramsey number results if the graphs appearing in the first argument are replaced by subgraphs with the same chromatic number. In some sense these graphs are the most general F for which large order trees are F-good. This follows from the following theorems.

<u>Theorem.</u> [10]

Let $\ell_1 \leq \ell_2 \ldots \leq \ell_m$ be fixed positive integers. If n is sufficiently large, then

$$r(K(1,\ell_1,\ell_2,\ldots,\ell_m),K(1,n)) = \begin{cases} m(\ell_1+n-2)+1 & \text{for } \ell_1 \text{ and } n \text{ even} \\ m(\ell_1+n-1)+1 & \text{otherwise.} \end{cases}$$

<u>Theorem.</u> [9]

For n sufficiently large
$r(K(2,2),K(1,n)) > n + n^{1/2} - 5n^{3/10}$

These results suggest that large order "tree-like" graphs might be F-good for an arbitrary fixed graph F, if these tree-like graphs do not contain vertices of large degree. The following result confirms this.

<u>Theorem.</u> [12]

Let G be a connected graph on n vertices and F a fixed graph on p vertices with chromatic number χ and chromatic surplus s. There exist positive constants ε_1 and ε_2 such that if n sufficiently large and both $q(G) \le n + \varepsilon_1 n^{1/(2p-1)}$ and $\Delta(G) \le \varepsilon_2 n^{1/(2p-1)}$, then

$$r(F,G) = (\chi-1)(n-1) + s.$$

This result says that G is F-good when G has limited degree and "essentially" n edges. The focus of this paper will be to show this edge condition can be weakened when F is a bipartite graph. In particular we prove the following theorem.

<u>Theorem 1.</u>

Let $\ell \le m$ be fixed positive integers and let G be a connected graph on n vertices. There exists a positive constant ε such that n sufficiently large and $\Delta(G) \le \varepsilon n^{1/(\ell+2)}$ imply that $r(K(\ell,m),G) = n + \ell - 1$.

The proof of Theorem 1 requires three more lemmas. The first is very simple and we omit the proof.

Lemma 2.

Let G be a graph of order n and maximum degree $\Delta(G) \leqq d$. Then G contains at least $n/(d^2+1)$ vertices such that the distance between any two of them is at least three.

It is helpful at this point to decide on a uniform notation to be used in the following two lemmas and in the proof of Theorem 1. Throughout, $G(V,E)$ will denote a graph of order n. We shall use $[S]^k$ to denote the collection of k-element subsets of a set S. Let $U = \{1,2,...,p\}$ and consider a two-coloring of $[U]^2$ using colors red and blue. The resulting monochromatic graphs will be denoted R and B respectively. In the proof of Theorem 1, we need to show that, subject to an appropriate growth condition on $\Delta(G)$, when $p = n + \ell - 1$ there is either an embedding of $K(\ell,m)$ into R or else an embedding of G into B. The following lemma gives a start toward an embedding of G into B.

Lemma 3.

Suppose that G has s vertices $x_1,...,x_s$ such that the distance between any two of them is at least three. Further, suppose that with $U = \{1,2,...,n\}$ and R and B as described above, excluding $\{n-s+1,...,n\}$, every vertex has degree at most M in R. Let X consist of $x_1,...,x_s$ together with their additional neighbors $x_{s+1},...,x_k$ in G. If

$$\delta(B) > M(\Delta(G) - 1) + s - 1$$

then there is a map $\rho: X \to U$ which is an embedding of $\langle X\rangle$ into B and where $\rho(x_j) = n-s+j$, $j = 1,\ldots,s$.

Proof. Define ρ one vertex at a time. For some $s < j \leqq k$ we can fail to find an appropriate $\rho(x_j)$ only if for the unique x_i, $i \leqq s$, to which x_j is adjacent in G, every vertex in the neighborhood of $\rho(x_i)$ in B is either an x_k, $k \neq i$, or else adjacent in R to one of the at most $\Delta(G)-1$ vertices which are images of neighbors of x_j in G. But these images have degree at most M in R. Thus, the stated inequality assures us that we do not fail.

The next lemma is a version of a result used by Sauer and Spencer in [14] and the proof technique is exactly as in the proof of their Theorem 3. It is a key result in the proof of our Theorem 1.

Lemma 4.

Let $G(V,E)$ be a graph of order n and let $U = \{1,..n\}$, R and B be as described above. If

$$n - k > 2\,\Delta(G)\,\Delta(R),$$

then given any $X \in [V]^k$ and any map $\rho: X \to U$ which is an embedding of $\langle X\rangle$ into B, ρ extends to a map $\sigma: V \to U$ which is an embedding of G into B.

Proof. Given any map $\sigma: V \to U$ which is an extension of ρ, let G_σ denote the corresponding image of G, i.e. $E(G_\sigma) = \{\sigma(uv): uv \in E(G)\}$. Of course, such a G_σ is not necessarily monochromatic. Let us denote by $E_R(G_\sigma)$ and $E_B(G_\sigma)$ the sets of red edges and blue edges respectively in G_σ. We claim that if the extension σ is chosen so that G_σ has as many blue edges as possible, then, in fact, $E_R(G_\sigma)$ will be empty. Suppose not, i.e. suppose that there is an edge $uv \in E(G)$ for which $\sigma(uv)$ is red. As σ yields an embedding of $\langle X \rangle$ into B, we may assume that $v \notin X$. We would like to do an "exchange" by introducing a new map $\tau: V \to U$ given by $\tau(v) = \sigma(w)$, $\tau(w) = \sigma(v)$ and $\tau = \sigma$ otherwise. For this purpose, a vertex $w \in V\backslash X$ is <u>bad</u> if any one of the following four conditions is satisfied: (i) $w = v$, (ii) $\sigma(vw) \in E_R(G_\sigma)$, (iii) for some $z \in V$, $vz \in E(G)$ and $\sigma(zw)$ is red, (iv) for some $z \in V$, $zw \in E(G)$ and $\sigma(vz)$ is red. Suppose that $\sigma(v)$ is of degree d in the red subgraph of G_σ. Then d vertices w satisfy (ii) and at most $d(\Delta(R) - 1) + (\Delta(G) - d)\Delta(R) = \Delta(G)\Delta(R) - d$ vertices satisfy (iii). Similarly, at most $\Delta(G)\Delta(R) - d$ vertices satisfy (iv). Since $d \geq 1$, there are at most $1 + d + 2(\Delta(G)\Delta(R)-d) \leq 2\Delta(G)\Delta(R)$ bad vertices. In view of the assumed inequality, there is a vertex $w \in V\backslash X$ which is a good choice for the exchange. Using (i)-(iv) it is easily checked that all of the edges incident with either $\tau(v)$ or $\tau(w)$ in G_τ are blue. Edges which are not incident with either $\tau(v)$ or $\tau(w)$ are not affected. In particular, τ is still an extension of ρ but G_τ has more blue edges than does G_σ. This contradiction of our choice of completes the proof.

The following slight strengthening of Lemma 4 will be the version actually used in the proof of Theorem 1. It is a corollary to the proof of the lemma.

Corollary.

Lemma 4 remains valid if the inequality $n - k > 2\Lambda(G)\Lambda(R)$ continues to hold then $\Lambda(R)$ is replaced by a bound M chosen so that no vertex x for which $\rho(x)$ is of degree $> M$ in R is adjacent to any vertex in $V \backslash X$.

Proof of Theorem 1. In view of Lemma 1, we need only show that if $U = \{1,2,\ldots,n + \ \ -1\}$ then in the two-coloring of $[U]^2$, either R will contain $K(\ell,m)$ or else B will contain G. Suppose that R contains no $K(\ell,m)$. Then an easy argument shows that in any collection of ℓ vertices at least one will have degree $\geqq \lceil (n-m)/\ell \rceil$ in B. Delete the $\ell - 1$ vertices of highest degree in R. For convenience, the vertices deleted are $n+1,\ldots,n+\ell-1$. Now let $U = \{1,2,\ldots,n\}$ and let R and B refer to the red and blue subgraphs of $[U]^2$. By the observation made earlier, we know that $\delta(B) > n/\ell - O(1)$. For an M yet to be chosen, let r denote the number of vertices which have degree at least M in R. Since there is no red $K(\ell,m)$ a standard argument yields the fact that

$$r \binom{M}{\ell} \leqq (m-1) \binom{n}{\ell}$$

and so

$$r < (m-1)(n/(M-\ell+1))$$

Set $a = 1/(\ell+2)$, $b = \ell/(\ell+2)$, $c = (\ell+1)/(\ell+2)$,

$$d(n) = c_1 n^a$$

and

$$M(n) = c_2 n^c.$$

Now the proof reduces to routine calculations. Setting

$$s(n) = \lceil n/(d^2 + 1) \rceil$$

and assuming that $\Delta(G) \leqq d(n)$, we can apply Lemma 3 if

$$s(n) \geqq r(n) \equiv (m-1)(n/(M-\ell+1))^{\ell}$$

and $$n/\ell - O(1) > Md + s.$$

We can then apply the corollary to Lemma 4 if

$$n - s(d+1) > 2dM.$$

For $\ell \geqq 2$ the desired inequalities hold for all sufficiently large n if

$$C_2^{\ell} / C_1^{2} > m - 1$$

and $$C_1 \, C_2 < 1/\ell.$$

Hence, Theorem 1 holds if we choose $\varepsilon < (m-1)^{-a} \, {}^{-b}$.
For $\ell = 1$ the theorem holds with $\varepsilon < (2(m-1))^{-1/3}$.

Theorem 1 holds in a slightly more general form when the complete bipartite graph is replaced by an arbitrary bipartite graph of order $\ell + m$ and with chromatic surplus ℓ.

Theorem 2.

Let F be a bipartite graph of order $\ell + m$ with chromatic surplus ℓ and let G be a connected graph of order n. There exists a positive constant ε such that $\Delta(G) \leqq \varepsilon \, n^{1/(\ell+2)}$ and n sufficiently large imply that G is F-good.

It should be noted that well-known Ramsey numbers for special pairs of graphs appear as corollaries to Theorems 1 and 2 or to the theorem of [12], at least when n is large. Some examples, for n large, are $r(K_m,C_n)$, $r(C_m,C_n)$, $r(C_m,P_n)$, $r(T_m,P_n)$, $r(K(\ell_1,\ell_2,\ldots,\ell_m),P_n)$, $r(K(\ell_1,\ell_2,\ldots,\ell_m),C_n)$. In addition if G_n is a regular connected graph of fixed degree, then for n large Theorem 1 gives $r(K(\ell,m),G_n) = n+\ell-1$ $(\ell \leq m)$. This would not follow from the earlier results.

The most natural question left unanswered involves the improvement of the results given in Theorems 1 and 2 and the theorem of [12]. Specifically, can the edge condition or the maximum degree condition in any of these theorems be weakened? It is likely that there exists a constant $c < 1$ such that these results hold for $\Delta(G) \leq cn$ and G of bounded edge density. Here edge density is defined as $\max_{H \leq G} q(H)/p(H)$.

In another direction, what about $r(F,G)$ when F is not bipartite? In [4] it was conjectured that if F is any fixed graph, then any sufficiently large connected graph with bounded degree is F-good. This attractive conjecture seems difficult, but in view of the results proved here, it may yield to a determined attack.

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