 CLIQUE NUMBERS OF GRAPHS 

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Received July 29, 1985
Revised October 11, 1985

For each natural number, denote by $G(n)$ the set of all numbers $c$ such that there exists a graph with exactly $c$ cliques (i.e., complete subgraphs) and $n$ vertices. We prove the asymptotic estimate

$$|G(n)| = o(2^n \cdot n^{-2/5})$$

and show that all natural numbers between $n + 1$ and $2^{n-6n^{4/6}}$ belong to $G(n)$. Thus we obtain

$$\inf |G(n)| > 0,$$

while

$$\lim \frac{|G(n)|}{a^n} = \infty \quad \text{for all } 0 < a < 2.$$

Many graph-theoretical problems involve the study of cliques, i.e., complete subgraphs (not necessarily maximal). In this context the following combinatorial problem arises naturally: For which numbers $n$ and $c$ is there a graph with $n$ vertices and exactly $c$ cliques? For fixed $n$, let $G(n)$ denote the set of all such ‘clique numbers’ $c$. Since each singleton and the empty set are always cliques, we have

$$n < c \leq 2^n \quad \text{for all } c \in G(n).$$

It is easy to check that every integer between $n + 1$ and $2^{n/2}$ occurs in $G(n)$ (see the remark at the end of this paper), and a more thorough investigation shows that even all integers between $n + 1$ and $2^{n/3}$ are clique numbers of suitable graphs with $n$ vertices. For small $n$, the first jumps in $G(n)$ occur between $2^{2n/3}$ and $2^{2n/3} \cdot 2$. Denoting by $c(n)$ the smallest $c > n + 1$ with $c \notin G(n)$, we obtain Table 1. (As usual, $[a]$ denotes the greatest integer not greater than $a$, while $\lfloor a \rfloor$ denotes the least integer not less than $a$.)

In the higher regions near $2^n$, $G(n)$ has large gaps. For example, the only clique numbers above $2^{n-1}$ are the numbers $2^{n-1} + 2^k$ with $0 \leq k < n$. The number $c_1$ of ones in the binary expansion of a given number $c$ plays a crucial role for the question whether $c$ is the clique number of a graph with $n$ vertices (see the proof of Theorem 1). As a consequence of the fact that $c_1$ cannot be too large for
c ∈ G(n), we show that the ratio |G(n)|/2^n tends to zero when n → ∞. But, on the other hand, it will turn out that for all positive reals a < 2, the ratio |G(n)|/a^n goes to infinity, and moreover, that all numbers c between n + 1 and 2^{n−6n/b} belong to G(n). In particular, for each b < 1 there is an n_b such that c(n) > 2^{bn} whenever n > n_b. Of course, this result disproves the conjecture (suggested by the above table) that c(n) would not exceed 2^{2n/3}.2.

In order to determine the sets G(n), it suffices to compute, for each natural number c, the smallest n such that there exists a graph with n vertices and c cliques. This is an immediate consequence of the following observation:

\[ c ∈ G(n) \text{ and } n + 1 < c \implies c ∈ G(n + 1). \]  

(1)

In fact, if G is a graph with n vertices and c > n + 1 cliques then G must have at least one edge joining two vertices, say, x and y. Delete this edge, adjoin a new vertex z to G, and join it with all vertices which are already joined with both, x and y. This gives a new graph G' with n + 1 vertices, but the number of cliques remains the same as for G because each clique of G containing x and y is replaced by a clique of G' containing z. (Cf. Fig. 1.)

Next, we derive an asymptotic upper bound for the cardinality of G(n):

**Theorem 1.** |G(n)| = \( o(2^n \cdot n^{-2/3}) \).

**Proof.** Let G be a graph with n vertices and c cliques. Choose a clique K of maximal size, say, k. Denoting by ℓ the set of all cliques of the induced subgraph G - K, we have

\[ c = \sum_{c ∈ ℓ} 2^{d_c}, \]

where \( d_c \) is the number of vertices in K joined with each vertex of C. By
maximality of $K$, $d_C$ is not greater than $k - |C|$, whence
\[ c \leq \sum_{j=0}^{n-k} \binom{n-k}{j} 2^{k-j} = \left( \frac{1}{2} \right)^{n-k} 2^n. \]

Furthermore, the number $c_1$ of ones in the binary expansion of $c$ is bounded by the cardinality of $\mathcal{C}$, whence
\[ c_1 \leq |\mathcal{C}| \leq 2^{n-k}. \]

Combining both inequalities, we obtain
\[ c \cdot c_1^\alpha \leq 2^n, \quad \text{where } \alpha = 2 - \log_2 3 > \frac{\beta}{2}. \]

Now choose an arbitrary real number $\beta$ with $\frac{\beta}{2} < \beta < \alpha$, and let
\[ m := \left\lfloor n - \beta \log_2 n + 1 \right\rfloor. \]

If $c \geq 2^m$, then $c_1 \leq 2^{(n-m)/\alpha} \leq 2^{(\beta/\alpha) \log_2 n} = n^{\beta/\alpha}$. Hence
\[ \left| \{c \in G(n): c \geq 2^m\} \right| \leq \left| \{c \in G(n): c_1 \leq n^{\beta/\alpha}\} \right| \leq \sum_{k=0}^{\left\lfloor \beta \log_2 n \right\rfloor} \binom{n}{k} \leq n^{1 + n^{\beta/\alpha}} = o(2^n \cdot n^{-2/5}) \quad \text{since } \beta/\alpha < 1. \]

On the other hand, we have
\[ \left| \{c \in G(n): c < 2^m\} \right| \leq 2^n - 2^{\beta \log_2 n + 1} = o(2^n \cdot n^{-2/5}) \quad \text{since } \beta > 2/5. \]

Table 2 suggests that $2^n \cdot n^{-2/5}$ is also a good estimate for small values of $|G(n)|$. Although $|G(n)|$ is of smaller order than $2^n$, we shall show in the second part of this paper that $\log_2 |G(n)|$ is asymptotically equal to $n$.

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<thead>
<tr>
<th>$n$</th>
<th>1</th>
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<th>3</th>
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<td>G(n)</td>
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<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
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<td>$[2^n \cdot n^{-2/5}]$</td>
<td>2</td>
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<td>16</td>
<td>31</td>
<td>58</td>
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Henceforth let $m$ be a natural number and
\[ s := m^{1/6}, \quad r := \left\lfloor \frac{s - 1}{2} \right\rfloor. \]

For any nonempty finite set $V$ of integers, put
\[ d(V) := \max V - \min V. \]

We shall use the following version of the 'pigeon-hole principle':

(PP) If $W$ is a set of $w$ integers, then for all natural numbers $v$ with $1 < v < w$ there exists a subset $V$ of $W$ with $v$ elements and $\left\lfloor (w-1)/(v-1) \right\rfloor d(V) \leq \left\lfloor \frac{s - 1}{2} \right\rfloor$.
For the construction of suitable graphs with prescribed clique numbers, we need a somewhat technical definition. Call a set $V$ of nonnegative integers $m$-adequate if the following conditions are satisfied (recall that $r$ and $s$ are functions of $m$) (cf. Fig. 2):

- $V = V_1 \cup V_2$ with max $V_1 < \min V_2$, \quad \mid V_1 \mid = r^2 + 1$ and \quad \mid V_2 \mid = 2r,$
- $m - \max V \geq s^5,$
- $d(V) \leq \frac{3}{2} s^5,$
- $d(V_i) \leq s^3,$
- $\min V_2 - \min V \geq \frac{1}{2} s^4.$

Our main result is prepared by an auxiliary lemma ensuring that there are enough $m$-adequate sets.

**Lemma.** Every set $W \subseteq \{0, \ldots, m - 1\}$ with not less than $2s^5$ elements contains an $m$-adequate set.

**Proof.** Choosing the $\lfloor s^5 \rfloor$ smallest elements from $W$, we obtain a subset $W_i$ with $d(W_i) \leq \max W_i \leq m - s^5$. Now (PP) gives a subset $W_2$ of $W_i$ with $\lfloor \frac{3}{2}s^4 \rfloor + 1$ elements such that $d(W_2) \leq \frac{3}{2} s^5$. In fact, $2s^5 \leq m = s^6$ implies $s \geq 2$, whence

$$d(W_i) \leq \frac{\lfloor \frac{3}{2}s^4 \rfloor}{s^5} - \frac{\lfloor \frac{3}{2}s^4 \rfloor - 1}{s^5} = \frac{3}{2} s^5.$$

The $\lfloor \frac{3}{2}s^4 \rfloor$ smallest elements of $W_2$ form a subset $W_3$. Again by (PP), we can select a subset $V_1$ of $W_3$ with $r^2 + 1$ elements and $d(V_1) \leq s^3$, because $s \geq 2$ and $r = \lfloor \frac{1}{2}(s - 1) \rfloor$ implies

$$d(W_2) \leq d(W_3) \leq \frac{r^2}{\lfloor \frac{1}{4}s^4 \rfloor} = \frac{s^2}{s^4 - s^2} \leq \frac{3s^7}{4s^4 - 4s^2} \leq s^3.$$
Finally, let $V_2$ consist of the $2r$ greatest elements of $W_2$ (cf. Fig. 3).

Then $W_2 \setminus V_2$ has $\left\lceil \frac{3s^4}{2} \right\rceil + 1 - 2r \geq \left\lceil \frac{3s^4}{2} \right\rceil$ elements (because $s \geq 2$ and $r \leq \frac{1}{2}(s - 1)$ yields $\left\lceil \frac{3s^4}{2} \right\rceil - 2r \geq \frac{3s^4}{2} - s + 1 \geq \frac{1}{2}s^4 + 1$). Thus $V_1 \subseteq W_3 \subseteq W_2 \setminus V_2$ and therefore max $V_1 \leq \text{min } W_3 < \min V_2$. Moreover,

$$\min V_2 - \min V_1 \geq \min V_2 - \max V_1 + r^2 \geq \min V_2 - \max W_3 + r^2 \geq |W_2 \setminus (W_3 \cup V_2)| + 1 + r^2 \geq \frac{3s^4}{2} - \frac{1}{2}s^4 - 2r + 1 + r^2 \geq \frac{1}{2}s^4 + (r - 1)^2 \geq \frac{3s^4}{2}.$$

Hence $V = V_1 \cup V_2$ has the required properties. □

Now we can prove

**Theorem 2.** For all natural numbers $n$ and $c$ with $n < c \leq 2^{n-6n^{16}}$ there is a graph with exactly $n$ vertices and $c$ cliques.

**Proof.** Let $c = 2^m + \sum_{d \in W} 2^d$, with $W \subseteq \{0, \ldots, m-1\}$. Furthermore, let $\mathcal{V}$ be a maximal collection of pairwise disjoint $m$-adequate subsets of $W$. By the lemma, the remainder $\tilde{W} = W \setminus \bigcup \mathcal{V}$ contains less than $2s^5$ elements where $s = m^{16}$. Now a graph $G$ with exactly $c$ cliques is constructed as follows. First, form an $m$-element clique $M$. Second, choose a family $\{G_V : V \in \mathcal{V}\}$ of pairwise disjoint $(2r+1)$-sets outside of $M$. Consider one such $G_V = \{x_1, \ldots, x_r, y_1, \ldots, y_r, z\}$ and make it a bipartite graph by joining each $x_i$ with each $y_i$. The $m$-adequate set $V = V_1 \cup V_2$ is labelled in form of an $(r+1) \times (r+1)$ array such that

$$V_1 = \{d_{00}\} \cup \{d_{ij} : 1 \leq i, j \leq r\} \quad (|V_1| = r^2 + 1),$$
$$V_2 = \{d_{0i} : 1 \leq i \leq r\} \cup \{d_{ij} : 1 \leq j \leq r\} \quad (|V_2| = 2r),$$
$$d_{00} < d_{ij} < d_{i-1,j} < d_{i-1,0} < d_{i,0} \quad (2 \leq i \leq r, 1 \leq j \leq r),$$
$$d_{00} < d_{ij} < d_{i,j-1} < d_{0,j-1} < d_{0j} \quad (1 \leq i \leq r, 2 \leq j \leq r).$$

Now we define an integer-valued $(r+1) \times (r+1)$ matrix $(s_{ij})$ by setting

$$s_{00} := d_{00},$$
$$s_{ij} := d_{ij} - d_{00} \quad (1 \leq i, j \leq r),$$
$$s_{10} := d_{10} - d_{00} - \sum_{j=1}^{r} (d_{ij} - d_{00}),$$

where $s_{ij}$ is the number of elements in the $i,j$-th cell of the matrix.
\[ s_{01} := d_{01} - d_{00} - \sum_{i=1}^{r} (d_{i1} - d_{00}), \]
\[ s_{i0} := d_{i0} - d_{i-1,0} + \sum_{j=1}^{r} (d_{i-1,j} - d_{ij}) \quad (2 \leq i \leq r), \]
\[ s_{0j} := d_{0j} - d_{0,j-1} + \sum_{i=1}^{r} (d_{i,j-1} - d_{ij}) \quad (2 \leq j \leq r). \]

Then we have
\[ s_{ij} \geq 0 \quad (0 \leq i, j \leq r). \] (1)

This is clear for \( i = j = 0 \) and for \( i + j > 1 \). By definition of \( m \)-adequate sets, we obtain
\[ s_{10} \geq \min V_2 - \min V - r \cdot d(V_1) \geq \frac{1}{2} s^4 - rs^3 > 0, \quad \text{since} \quad r < \frac{1}{2} s. \]

The same inequality holds for \( s_{01} \). Next, one proves by induction
\[ d_{i0} = d_{00} + \sum_{k=1}^{i} s_{k0} + \sum_{j=1}^{r} s_{ij} \quad (1 \leq i \leq r), \] (2)
\[ d_{j0} = d_{00} + \sum_{k=1}^{j} s_{0k} + \sum_{i=1}^{r} s_{ij} \quad (1 \leq j \leq r). \]

Third, we have the inequality
\[ \sum_{i=0}^{r} \sum_{j=0}^{r} s_{ij} \leq m. \] (3)

In fact,
\[
\begin{align*}
\sum_{i=0}^{r} \sum_{j=1}^{r} s_{ij} &+ \sum_{i=1}^{r} \sum_{j=1}^{r} s_{ij} + \sum_{i=1}^{r} s_{i0} + \sum_{j=1}^{r} s_{0j} = (2) \\
&= d_{00} + \sum_{i=1}^{r} \sum_{j=1}^{r} s_{ij} + d_{00} - d_{00} - \sum_{j=1}^{r} s_{ij} + d_{00} - d_{00} - \sum_{i=1}^{r} s_{ir} \\
&= \sum_{i=1}^{r-1} \sum_{j=1}^{r} (d_{ij} - d_{00}) + (d_{00} - d_{rr}) + d_{0r} \\
&\leq (r - 1)^2 d(V_1) + d(V) + \max V \\
&\leq \frac{s^2}{4} + \frac{3}{2} s^2 + m - s^3 = m.
\end{align*}
\]

On account of (1) and (3), we can choose a family of pairwise disjoint subsets \( S_{ij} \) (cf. Fig. 4) of \( M \) with \( s_{ij} \) elements \( (0 \leq i, j \leq r) \). Join \( x_i \) with all points of the set
\[ X_i = \bigcup_{k=0}^{r} S_{k0} \cup \bigcup_{j=1}^{r} S_{ij} \quad (1 \leq i \leq r), \]
and join $y_j$ with all points of the set

$$Y_j = \bigcup_{k=0}^{j} S_{0k} \cup \bigcup_{i=1}^{r} S_{ij} \quad (1 \leq j \leq r).$$

By (2), we have

$$|X_i| = d_{i0} \quad (1 \leq i \leq r),$$

$$|Y_j| = d_{0j} \quad (1 \leq j \leq r).$$

Furthermore, the number of points joined with both, $x_i$ and $y_j$, is

$$|S_{00} \cup S_{ij}| = d_{00} + s_{ij} = d_{ij} \quad (1 \leq i, j \leq r).$$

Finally, join the remaining point $z$ of $G_V$ with the points of $S_{00}$ and recall that $|S_{00}| = d_{00}$. Then the number of cliques containing at least one point from $G_V$ amounts to

$$\sum_{i=0}^{r} \sum_{j=0}^{r} 2^{d_{ij}} = \sum_{d \in V} 2^d.$$
After having carried through this procedure for each $V \in \mathcal{V}$, choose for each of the remaining exponents $d \in \hat{W} = W \setminus \mathcal{V}$ a new point and join it with exactly $d$ points of $M$. The graph obtained in this way has precisely $c = 2^m + \sum_{d \in \hat{W}} 2^d$ cliques, and the number of vertices is

$$m + (2r + 1) \cdot |\mathcal{V}| + |\hat{W}| < m + \frac{2r + 1}{(r + 1)^2} |\hat{W}| + 2s^5 \leq m + \frac{4sm}{s^2} + 2s^5 = m + 6m^{5/6}.$$  

(For the last inequality, observe that $|\hat{W}| \leq m$ and $r = \lfloor \frac{1}{2}(s - 1) \rfloor$.) Now $c \leq 2^{n-6n^{5/6}}$ implies

$$m = \lfloor \log_2 c \rfloor \leq n - 6n^{5/6} \quad \text{whence} \quad m + 6m^{5/6} \leq n.$$  

But by our introductory remark (*), $n < c \in G(n')$ for some $n' \leq n$ implies $c \in G(n)$, and the proof is complete. \hfill \Box

Of course, for small values of $n$ the statement of Theorem 2 is much weaker than the implication

$$n < c \leq 2^{n/2 + 1} \Rightarrow c \in G(n),$$

which follows by induction from the obvious implication

$$c \in G(n) \Rightarrow c + 1 \in G(n + 1) \quad \text{and} \quad 2c \in G(n + 1).$$

As an immediate consequence of Theorems 1 and 2, we finally notice:

**Corollary.**

$$\lim_{n \to \infty} \frac{|G(n)|}{2^n} = 0, \quad \text{but} \quad \lim_{n \to \infty} \frac{|G(n)|}{a^n} = \infty \quad \text{for} \ 0 < a < 2.$$